A Meaning Based Information Theory - Informalogical Space: Basic Concepts and Convergence of Information Sequences

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Abstract

This paper presents an information theory that is based on meanings and relationships of information. This paper first introduces the very basic concept in our approach, a binary relation *contain* between two pieces of information, which is based on inference between the two pieces of information. Then, based on the contain relation, this paper introduces two basic operations *union* and *intersection* on a collection (*i.e.*, set) of information.

This paper lays the basis of our approach by introducing the core concept, *informalogical space*. An informalogical space is a collection of information that satisfies certain conditions represented in terms of the contain relation, and the union and intersection operations. An informalogical space is similar to a topological space in symbolic sense, but is different in nature.

This paper also introduces *information net* in an informalogical space. Information net is a generalization of *information sequence*, just like that net is a generalization of sequence in general topology. This paper will build a *convergence* theory of information net that is similar to the Moore-Smith convergence in general topology in symbolic sense. Then, this paper applies the results on information nets to information sequences.

Key words:
information, informalogy, informalogical space, accumulation information, information net, convergence, limit information, cluster information, information sequence.

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1 Introduction

We are in an era of information explosion, especially after the introduction of the Internet. To tackle the information explosion, various efforts have been made to expend the theoretical capacity and application areas of information theory. (See [3,4,8,9,13,28].) A lot of work has also been done in the related fields of artificial intelligence and knowledge discovery. (See [24,25,27,29,31–33,36,38,39,42].) In these fields, a lot of contributions have been made to machine learning or computational learning. (See [1,2,5,6,17,19,21,24,28,31–34].)

Another related and active field is soft computing. Instead of computing with numbers, a lot of works have been done to tackle the problem of computing with words, meaning, inference and reasoning based on meaning. (See [7,12,14,28,30,35,40,41,49–51].)

In this paper, we will present a theory of information that is based on meanings and relationships of information. In the approach of the current work, no quantitative measurement is introduced. Instead, the meanings and relationships of information play a central role in the approach of the current work, such as in judging whether an information sequence converges to some piece of information.

A basic concept in this approach is the binary relation *contain* between two pieces of information. For example, information J contains information I if we can infer I from J. Thus, contain relation is based on inference between two pieces of information. In practice, if both J and I are relatively small pieces of information, then it may well be the case that we cannot infer I from J alone. However, if an extra piece of information, say K presents, then we can infer I from J and K combined together. Then, the combination of J and K can be a new piece of information, L, that contains the information I.

Based on the contain relation, we will introduce two operations on information sets: *union* and *intersection*. For example, the union of information I and information J is the sum of the two pieces of information, and the intersection of information I and information J is the common information that is contained in each of the two pieces of information.

The core concept of our theory is *informalogical space*. An informalogical space is a collection (*i.e.*, set) of information that satisfies two conditions represented in terms of the binary contain relation, and union and intersection operations. An informalogical space is similar to a topological space in symbolic structure, but is different from a topological space in nature. Once the concept of informalogical space is introduced, our discussions will be performed in informalogical spaces.

Based on the contain relation, we can introduce the local structure in an informalogical space, which is the *neighborhood* system of a piece of information in an informalogical space. A neighborhood of a piece of information I in an informalogical space contains the information which, judged from the structure of the informalogical space, are "close" to I with respect to meaning.

After we establish the concept of neighborhood of a piece of information in an informalogical space, we will be able to naturally introduce the convergence and limit information of information sequences. Actually, we will start with establishment of the convergence theory of *information net* in an informalogical space. Information net is a generalization of information sequence, just like that net is a generalization of sequence in general topology. Our convergence theory of information net is similar to the Moore-Smith convergence in general topology in symbolic and terminological sense, but different in nature, since the notions of "neighborhood" are different in nature. We will apply the results that we obtain on information nets to information sequences.

In Section 2, we will first introduce a binary relation *contain* between two pieces of information. Based on the contain relation, we will introduce two operations, *union* and *intersection*, on a collection of information. Once the contain relation, and union and intersection operations are introduced, we will be able to introduce the core concept of our information theory, *informalogical space*. We will discuss the similarities in symbolic structure and differences in nature between a topological space and an informalogical space. All our subsequent discussions, including those of *information net*, will be performed in an informalogical space.

In Section 3, we will build the local structure in an informalogical space, which is the *neighborhood* system of a piece of information in an informalogical space. We will first introduce the concept of *information interval* in an informalogical space and prove some properties about information intervals. Then, with the use of information interval, we will introduce the concept of neighborhood of a piece of information in an informalogical space. We will prove a property of neighborhood and introduce the concept of *accumulation information* of an information set.

In Section 4, we will introduce the concept of *information net*, which is a generalization of information sequence. We will discuss the *convergence* of an information net in an informalogical space. We will also introduce a particular type of informalogical space, *separated* informalogical space. We will prove that an informalogical space is separated if and only if every information net in it has at most one piece of limit information. We will also prove that a piece of information I is a piece of accumulation information of an information set $\mathcal{A}$ if and only if there exists an information net in $\mathcal{A} \setminus \{I\}$ that converges to I.

In Section 5, we will introduce the concept of *information subnet* and *cluster information* of an information net. We will prove that a piece of information I is a piece of cluster information of an information net if and only if the information net has a subnet that converges to I.

In Section 6, we will discuss a special type of information net, *information sequence*. We will also introduce a particular type of informalogical space, *first countable* informalogical space. A first countable informalogical space has good properties for information sequences.

In Section 7, we will briefly discuss some of our future works.

2 Basic Concepts and Informalogical Spaces

Information are things that can be used to eliminate or reduce uncertainty. In this paper, we use an upper-case letter such as I, J, X, Y, ... to represent a piece of information. We use an upper-case calligraphic letter such as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}, \mathcal{U}$, ... to represent an information set (*i.e.*, a collection of information). The things that can not be used to eliminate or reduce uncertainty will be called *zero information*, and we use 0 will be used to represent both the collective zero information and a piece of zero information.

We will introduce two operations on an information set, union and intersection. To make the operations meaningful, we need consistency of information: assume that I and J are two pieces of information. We say that I and J have consistency, and we call them consistent information if we cannot infer any contradictions from the two pieces of information.

In general, let $\mathcal{A}$ be a non-void information set. We say that $\mathcal{A}$ has *consistency*, and we call it a *consistent information set* if we cannot infer any contradictions from all the information in $\mathcal{A}$. If $\mathcal{A}$ is empty, we stipulate that it has consistency.

For example, information $I =$ "Mr. X is 36 years old." and information $J =$ "Mr. X is a software engineer." are two pieces of consistent information, whereas information $I =$ "Mr. Y is 5 feet and 8 inches" and information $J =$ "Mr. Y is 6 feet and 2 inches" are not two pieces of consistent information.

It should be pointed out that here, and in the entire paper, fuzziness is not taken into consideration. For example, one can have in the fuzzy sense using the above age example: "Mr. Y is short." and "Mr. Y is tall." If Mr. Y is 5'11", then his tallness (or, shortness) metric may belong equally to both sets' assertions. In other words, contradiction and consistence can be in degrees.

The current work is based on regular (or, non-fuzzy) set theory (see [18,20,22]). It is a limitation of the current work that it does not take fuzziness into consideration. Future works should consider fuzziness to generate more general approaches. (For fuzzy set and logic, see [15,16,23,26,37,44–49].)

With the limitation in mind, from now on, we assume that all the information under consideration are consistent information.

Definition 2.1 *Suppose I and J are two pieces of information. If I can be inferred from J, then we say that information I is contained in information J, and we also say that information J contains information I. We use $I \preceq J$ or $J \succeq I$ to represent this relation. We stipulate that a piece of zero information is contained in any information.*

We say that information I and information J are equal to each other if $I \preceq J$ and $J \preceq I$. We use $I = J$ to represent the equality. We write $I \neq J$ when information I and information J are not equal.

We say that information I is a piece of subinformation of information J if $I \preceq J$. We say that information I is a piece of proper subinformation of information J if $I \preceq J$ and $I \neq J$.

For an example of the contain relation, information $I =$ "There is a table in the room." is contained in information $J =$ "There is a black table in the room." However, for generally given two pieces of information I and J, it may well happen that neither I is contained in J nor J is contained in I.

It should be noted that this paper assumes the "inference" applied in Definition 2.1 has reflexivity and transitivity. In other words, we assume that the following two assumptions are true.

Assumption 1: I can be inferred from I itself.
Assumption 2: If I can be inferred from J, and J can be inferred from K, then I can be inferred from K.

If one of the above two assumptions is not true for a particular type of "inference", then the discussions of this paper may not apply to cases where that particular type of "inference" is applied, and in those cases, different and potentially more general discussions need to be formulated. Again, in this paper, we assume that the above two assumptions are true.

We can have the following theorem by translating the above two assumptions into the language of Definition 2.1.

Theorem 2.1 *Suppose that I, J and K are pieces of information. Then,*

(1) $I \preceq I$*; and*
(2) If $I \preceq J$ *and* $J \preceq K$*, then* $I \preceq K$*.*

Next, we introduce two basic operations on information, union and intersection.

Definition 2.2 *Let* $\mathcal{A}$ *be a non-void information set, i.e., the members of* $\mathcal{A}$ *are pieces of information. We use* $\vee\mathcal{A}$ *or* $\vee\{A|A \in \mathcal{A}\}$ *to represent the smallest one out of all the information that contain every member of* $\mathcal{A}$*. We call* $\vee\mathcal{A}$ *the union of all the pieces of information in* $\mathcal{A}$*, or the union of* $\mathcal{A}$ *for short. We use* $\wedge\mathcal{A}$ *or* $\wedge\{A|A \in \mathcal{A}\}$ *to represent the largest one out of all the information that are contained in every member of* $\mathcal{A}$*. We call* $\wedge\mathcal{A}$ *the intersection of all the pieces of information in* $\mathcal{A}$*, or the intersection of* $\mathcal{A}$ *for short. If* $\mathcal{A}$ *is a void set, we define that the union and intersection of* $\mathcal{A}$ *are zero information.*

An explanation of $\vee\mathcal{A}$ is: first, $A \preceq \vee\mathcal{A}$ for every $A \in \mathcal{A}$; and second, if there is another piece of information I such that $A \preceq I$ for every $A \in \mathcal{A}$, then $\vee\mathcal{A} \preceq I$. Similarly, an explanation of $\wedge\mathcal{A}$ is: first, $\wedge\mathcal{A} \preceq A$ for every $A \in \mathcal{A}$; and second, if there is another piece of information I such that $I \preceq A$ for every $A \in \mathcal{A}$, then $I \preceq \wedge\mathcal{A}$.

It is obvious that $\vee\mathcal{A}$ is the sum of all the pieces of information in the information set $\mathcal{A}$, and $\wedge\mathcal{A}$ is the common information that is contained in every piece of information in the information set $\mathcal{A}$. For example, $I =$ "XYZ is a poetess." and $J =$ "XYZ is a female politician." are two pieces of information. The union of I and J is $I \vee J =$ "XYZ is a poetess and a female politician.", and the intersection of I and J is $I \wedge J =$ "XYZ is a woman."

We have introduced the contain relation between two pieces of information, and the union and intersection operations on a set of information. The following theorem lists two basic properties that are obvious from the definitions.

Theorem 2.2 *Suppose that* A*,* B *and* X *are pieces of information. Then,*

(1) If $A \preceq X$ *and* $B \preceq X$*, then* $A \vee B \preceq X$*; and*
(2) If $X \preceq A$ *and* $X \preceq B$*, then* $X \preceq A \wedge B$*.*

Next, we will introduce the core concept of our approach, informalogical space.

Definition 2.3 *Let* $\mathcal{S}$ *be a non-void information set, and let* $\Omega = \vee\mathcal{S}$ *(i.e.,* Ω *is the union of all the information in* $\mathcal{S}$*). Let* $\mathcal{I}$ *be a non-void subset of* $\mathcal{S}$ *such that* $\vee\mathcal{I} = \Omega$*. We say that* $\mathcal{I}$ *is an informalogy, that* $\mathcal{S}$ *is the space of the informalogy* $\mathcal{I}$*, that* $\mathcal{I}$ *is an informalogy for the space* $\mathcal{S}$ *and that the pair* $(\mathcal{S}, \mathcal{I})$ *is an informalogical space, if the following two conditions hold:*

(1) if $I, J \in \mathcal{I}$, then $I \wedge J \in \mathcal{I}$; and
(2) if $\mathcal{I}_0 \subseteq \mathcal{I}$, then $\vee \mathcal{I}_0 \in \mathcal{I}$.

In general topology, topology and topological space are defined as the following: "A **topology** is a family $\mathcal{T}$ of sets which satisfies the two conditions: the intersection of any two members of $\mathcal{T}$ is a member of $\mathcal{T}$, and the union of the members of each subfamily of $\mathcal{T}$ is a member of $\mathcal{T}$. The set $X = \cup\{U : U \in \mathcal{T}\}$ is necessarily a member of $\mathcal{T}$ because $\mathcal{T}$ is a subfamily of itself, and every member of $\mathcal{T}$ is a subset of X. The set X is called the **space** of the topology $\mathcal{T}$ and $\mathcal{T}$ is a **topology for** X. The pair $(X, \mathcal{T})$ is a **topological space**." ([22, p. 37].)

If we translate symbolically the two conditions that the family $\mathcal{T}$ of sets must satisfy to be a topology, then they are:

(1) if $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$; and
(2) if $\mathcal{T}_0 \subseteq \mathcal{T}$, then $\cup \mathcal{T}_0 \in \mathcal{T}$.

In a symbolic sense, these two conditions are similar to the two conditions in our Definition 2.3. However, topological space and informalogical space carry different meanings and natures.

First, the union and intersection of sets and the union and intersection of information have different natures. That is why we used different symbols $\vee$ and $\wedge$ for the union and intersection of information. The reason why we did not use new terminologies is that "union" and "intersection" naturally reflect the two operations that we defined on information.

Second, the entities in a topological space and the entities in an informalogical space have different levels of structure. In a topological space $(X, \mathcal{T})$, $\mathcal{T}$ is a higher level structure on X (*i.e.*, $\mathcal{T}$ is a family of subsets of X). However, in an informalogical space $(\mathcal{S}, \mathcal{I})$, $\mathcal{I}$ has the same level of structure as $\mathcal{S}$ (*i.e.*, $\mathcal{I}$ is simply a subset of $\mathcal{S}$). We can discuss the intersection of two pieces of information in $\mathcal{S}$. However, it does not make sense to discuss the intersection of two points in X.

Third, in a topological space $(X, \mathcal{T})$, the points in X are the smallest elements. However, in an informalogical space $(\mathcal{S}, \mathcal{I})$, for the information in $\mathcal{S}$, due to the nature of information, we can discuss decompositions of the information. Actually, decomposition of information can be very useful in applications, and we will discuss decompositions of information in future works. General topology also has the terminology "decomposition", but it carries different meaning.

Finally, points in X have no relationship between one another. However, in $\mathcal{S}$, information have relationships between one another, like one piece of information is contained in another piece of information, and like the intersection of two pieces of information is non-zero, or the intersection is equal to another piece of information. In fact, the relationships among information in $\mathcal{S}$ is a major characteristic based on which an informalogical space is built.

Because of symbolic similarities and terminologies that we borrow from general topology, with the development of our theory of informalogical space, there will be concepts and results that have mirror items in general topology in a symbolic sense, even though they carry different meanings. For example, in general topology, "net" is a generalization of sequence. In this paper, we will borrow the term "net" and introduce "information net," which is a generalization of information sequence. Of course, because of the difference in nature, there will also be concepts and results that do not have mirror items in general topology even in symbolic sense.

We can see from Definition 2.3 that, when the informalogical space $(\mathcal{S}, \mathcal{I})$ is given, the information $\Omega = \vee\mathcal{S}$ is known. We give Ω a special name, *the reference information for the space* $\mathcal{S}$. In future, we may need to refer to Ω when we introduce some concepts. At that time, we may extend the notation $(\mathcal{S}, \mathcal{I})$ to $(\Omega, \mathcal{S}, \mathcal{I})$. However, in this paper, $(\mathcal{S}, \mathcal{I})$ is enough, and we will use it throughout the paper.

From now on, our discussions will be exclusively in an informalogical space. It is clear from the definition that the space $\mathcal{S}$ is meant to include all the pieces of information under discussion. We will use both *a piece of information in the space* and simply *a piece of information* to represent a member of $\mathcal{S}$. We will also use both *an information set in the space* and simply *an information set* to represent a subset of $\mathcal{S}$. It should be noted that a piece of information in the space may not be a member of the informalogy $\mathcal{I}$, and an information set in the space may not be a subset of the informalogy $\mathcal{I}$.

It is easy to see that for the same space, we can have different informalogies, and for the same informalogy, we can have different spaces.

The first condition in Definition 2.3 means that the intersection of two members of the informalogy is still a member of the informalogy. Actually, inferred from this, we know that the intersection of any finite number of members of the informalogy is still a member of the informalogy. The second condition in Definition 2.3 means that the union of any subset of an informalogy is still a member of the informalogy. This obviously implies that the union of any finite number of members of the informalogy is still a member of the informalogy. Furthermore, the void set ϕ and the informalogy $\mathcal{I}$ itself are two subsets of $\mathcal{I}$, and we know that $\vee\phi = 0$ and $\vee\mathcal{I} = \Omega$. Thus, any informalogy contains 0, the zero information, and Ω, the reference information for the space $\mathcal{S}$. We know that $\mathcal{I}$ is a subset of $\mathcal{S}$. Therefore, the space $\mathcal{S}$ must also contain 0 and Ω.

Definition 2.4 *Let $\mathcal{S}$ be any non-void information set that contains* 0 *and* Ω*, where* $\Omega = \vee\mathcal{S}$*. The smallest possible informalogy for the space* $\mathcal{S}$ *is* $\mathcal{I} = \{0, \Omega\}$*. This informalogy* $\mathcal{I}$ *is called the trivial informalogy for the space* $\mathcal{S}$*.*

3 Neighborhoods of A Piece of Information

In this section, we will build the local structure in an informalogical space, which is the neighborhood system of a piece of information in the space. First, we will introduce the concept of information interval and prove some properties of information intervals. Then, we will define neighborhood of a piece of information in the space. Finally, we will introduce accumulation information of an information set in the space.

Definition 3.1 *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space. Let* X *and* Y *be two members of the informalogy* $\mathcal{I}$*. We denote* $[X, Y] \equiv \{I | I \in \mathcal{S}$ *and* $X \preceq I \preceq Y\}$*.* $[X, Y]$ *is an information set which contains all the information in* $\mathcal{S}$ *that range from the lower endpoint* X *to the upper endpoint* Y*. We call* $[X, Y]$ *an information interval in the informalogical space* $(\mathcal{S}, \mathcal{I})$*, or simply call it an interval. When* $[X, Y]$ *is non-void, we call it a non-void interval; when* $[X, Y]$ *is void, we call it a void interval, and we use* θ *to denote a void interval.*

When both $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *are information intervals, and* $[X_1, Y_1] \subseteq [X_2, Y_2]$*, we say that* $[X_1, Y_1]$ *is a subinterval of* $[X_2, Y_2]$*.*

A set of intervals is called a family of intervals, or an interval family. We often use $\mathcal{U}$ *to represent an interval family.*

In case where no confusion is likely to result, we may simply use a single letter like U, V, etc. to represent an information interval. However, it should be kept in mind that an information interval is not a single piece of information, but a set of information.

Lemma 3.1 *Suppose that* $[X, Y]$*,* $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *are information intervals in the informalogical space* $(\mathcal{S}, \mathcal{I})$*. Then,*

(1) $[X, Y]$ *is non-void if and only if* $X \preceq Y$*;*
(2) $[X, Y]$ *is non-void if and only if* $X, Y \in [X, Y]$*;*
(3) if $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *are two non-void intervals, then,* $[X_1, Y_1] = [X_2, Y_2]$ *if and only if* $X_1 = X_2$ *and* $Y_1 = Y_2$*; and*
(4) if $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *are two non-void intervals, then,* $[X_1, Y_1] \subseteq [X_2, Y_2]$ *if and only if* $X_2 \preceq X_1 \preceq Y_1 \preceq Y_2$*.*

Proof

(1) When $[X, Y]$ is non-void, by Definition 3.1, there is $I \in \mathcal{S}$ such that $X \preceq I \preceq Y$. By 2 of Theorem 2.1, $X \preceq Y$. When $X \preceq Y$, by 1 of Theorem 2.1, we have $X \preceq X \preceq Y$, and thus $X \in [X, Y]$ since $X \in \mathcal{S}$. Similarly, we have $Y \in [X, Y]$. This means that $[X, Y]$ is non-void.
(2) When $[X, Y]$ is non-void, then, by 1, $X \preceq Y$. Then, by 1 of Theorem 2.1, $X \preceq X \preceq Y$ and $X \preceq Y \preceq Y$. Thus, $X, Y \in [X, Y]$ since $X, Y \in \mathcal{S}$. When $X, Y \in [X, Y]$, $[X, Y]$ is obviously non-void.
(3) Since $[X_1, Y_1]$ and $[X_2, Y_2]$ are non-void, by 2, $X_1, Y_1 \in [X_1, Y_1]$ and $X_2, Y_2 \in [X_2, Y_2]$.
 When $[X_1, Y_1] = [X_2, Y_2]$, we will have $X_1, Y_1 \in [X_2, Y_2]$. Thus, $X_2 \preceq X_1$ and $Y_1 \preceq Y_2$. Similarly, we have $X_2, Y_2 \in [X_1, Y_1]$ that implies $X_1 \preceq X_2$ and $Y_2 \preceq Y_1$. Therefore, $X_1 = X_2$ and $Y_1 = Y_2$.
 When $X_1 = X_2$ and $Y_1 = Y_2$, it is obvious that $[X_1, Y_1] = [X_2, Y_2]$.
(4) Since $[X_1, Y_1]$ is non-void, by 1 and 2, $X_1 \preceq Y_1$, and $X_1, Y_1 \in [X_1, Y_1]$.
 When $[X_1, Y_1] \subseteq [X_2, Y_2]$, we have $X_1, Y_1 \in [X_2, Y_2]$. Thus, $X_2 \preceq X_1 \preceq Y_2$ and $X_2 \preceq Y_1 \preceq Y_2$. By 2 of Theorem 2.1, $X_2 \preceq X_1 \preceq Y_1 \preceq Y_2$.
 When $X_2 \preceq X_1 \preceq Y_1 \preceq Y_2$, for each $I \in [X_1, Y_1]$, we have $X_1 \preceq I \preceq Y_1$. Then, by 2 of Theorem 2.1, we have $X_2 \preceq I \preceq Y_2$ since $X_2 \preceq X_1$ and $Y_1 \preceq Y_2$. That is, for each $I \in [X_1, Y_1]$, we have $I \in [X_2, Y_2]$. Thus, $[X_1, Y_1] \subseteq [X_2, Y_2]$.

□

We know from the definition of information interval that an interval is a special information set, *i.e.*, all the information in $\mathcal{S}$ ranging from the piece of lower endpoint information to the piece of upper endpoint information. It is obvious that not every information set can be an information interval.

Now, suppose $[X_1, Y_1]$ and $[X_2, Y_2]$ are two information intervals. As two information sets, they have an intersection set $[X_1, Y_1] \cap [X_2, Y_2]$. The following theorem will show that $[X_1, Y_1] \cap [X_2, Y_2]$ is also an information interval.

Theorem 3.1 *Suppose* $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *are two information intervals. Then, as two information sets, their intersection set* $[X_1, Y_1] \cap [X_2, Y_2]$ *is also an information interval, and*

$$[X_1, Y_1] \cap [X_2, Y_2] = [X_1 \vee X_2, Y_1 \wedge Y_2].$$

Proof It is obvious that $X_1 \vee X_2$ and $Y_1 \wedge Y_2$ are two members of the informalogy, since X_1, X_2, Y_1 and Y_2 are all members of the informalogy.

(1) Suppose $X_1 \vee X_2 \preceq Y_1 \wedge Y_2$ is not true. Then, $[X_1 \vee X_2, Y_1 \wedge Y_2]$ is a void set. On the other hand, we show $[X_1, Y_1] \cap [X_2, Y_2]$ is also a void set. Otherwise, there is a piece of information I such that $I \in [X_1, Y_1] \cap [X_2, Y_2]$. That

means, $X_1 \preceq I \preceq Y_1$ and $X_2 \preceq I \preceq Y_2$, and by 1 and 2 of Theorem 2.2, $X_1 \vee X_2 \preceq I \preceq Y_1 \wedge Y_2$. Then, by 2 of Theorem 2.1, $X_1 \vee X_2 \preceq Y_1 \wedge Y_2$. However, this is contradictory to the assumption that $X_1 \vee X_2 \preceq Y_1 \wedge Y_2$ is not true. Thus, $[X_1, Y_1] \cap [X_2, Y_2]$ is also a void set.

(2) Suppose $X_1 \vee X_2 \preceq Y_1 \wedge Y_2$ is true. Then by Lemma 3.1, $[X_1 \vee X_2, Y_1 \wedge Y_2]$ is a non-void information interval. Let $I \in [X_1 \vee X_2, Y_1 \wedge Y_2]$. Then, $X_1 \vee X_2 \preceq I \preceq Y_1 \wedge Y_2$, and by Definition 2.2, $X_1 \preceq I \preceq Y_1$ and $X_2 \preceq I \preceq Y_2$. That is, $I \in [X_1, Y_1]$ and $I \in [X_2, Y_2]$, which implies $I \in [X_1, Y_1] \cap [X_2, Y_2]$. In other words, we proved that $[X_1, Y_1] \cap [X_2, Y_2]$ is a non-void set, and $[X_1 \vee X_2, Y_1 \wedge Y_2] \subseteq [X_1, Y_1] \cap [X_2, Y_2]$. Next, we show $[X_1, Y_1] \cap [X_2, Y_2] \subseteq [X_1 \vee X_2, Y_1 \wedge Y_2]$. Let $J \in [X_1, Y_1] \cap [X_2, Y_2]$. Then, $J \in [X_1, Y_1]$ and $J \in [X_2, Y_2]$, which implies $J \in \mathcal{S}$, $X_1 \preceq J \preceq Y_1$ and $X_2 \preceq J \preceq Y_2$. Thus, by 1 and 2 of Theorem 2.2, $J \in \mathcal{S}$ and $X_1 \vee X_2 \preceq J \preceq Y_1 \wedge Y_2$, which means $J \in [X_1 \vee X_2, Y_1 \wedge Y_2]$. Thus, $[X_1, Y_1] \cap [X_2, Y_2] \subseteq [X_1 \vee X_2, Y_1 \wedge Y_2]$. Combining the above, we have $[X_1, Y_1] \cap [X_2, Y_2] = [X_1 \vee X_2, Y_1 \wedge Y_2]$ is a non-void interval.

□

With Theorem 3.1 established, we can introduce the following definition of intersection interval.

Definition 3.2 *Suppose* $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *are two information intervals. We say that the interval* $[X_1, Y_1] \cap [X_2, Y_2] = [X_1 \vee X_2, Y_1 \wedge Y_2]$ *is the intersection interval of* $[X_1, Y_1]$ *and* $[X_2, Y_2]$*, or simply call it their intersection.*

When $[X_1, Y_1] \cap [X_2, Y_2] = \theta$*, we say that the intervals* $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *are disjoint with each other; when* $[X_1, Y_1] \cap [X_2, Y_2] \neq \theta$*, we say that the intervals* $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *intersect each other.*

In fact, by repeatedly applying Theorem 3.1, the the statement of Theorem 3.1 can be generalized to any finite number of intervals. Thus, the definition of intersection interval can be generalized to any finite number of intervals. For a finite number of intervals $[X_1, Y_1]$, $[X_2, Y_2]$, ..., $[X_n, Y_n]$, their intersection interval is $\cap_{i=1}^{n}[X_i, Y_i] = [\vee_{i=1}^{n} X_i, \wedge_{i=1}^{n} Y_i]$. However, it should also be aware that the statement of Theorem 3.1 cannot be generalized to for an infinite number of intervals, since for infinite number of intervals, say, $[X_1, Y_1]$, $[X_2, Y_2]$, ..., $[X_n, Y_n]$, ..., the intersection $\wedge_{i=1}^{\infty} Y_i$ may not be a member of the informalogy $\mathcal{I}$, and thus $[\vee_{i=1}^{\infty} X_i, \wedge_{i=1}^{\infty} Y_i]$ may not be an information interval in the informalogical space.

As we mentioned before, sometimes we simply use a single letter, like U to denote an information interval.

Lemma 3.2 *Suppose* U*,* U_1*,* U_2 *and* U_3 *are information intervals. Then*

(1) $U \subseteq U$;
(2) If $U_1 \subseteq U_2$ and $U_2 \subseteq U_1$, then $U_1 = U_2$;
(3) If $U_1 \subseteq U_2$ and $U_2 \subseteq U_3$, then $U_1 \subseteq U_3$;
(4) $U_1 \cap U_2 \subseteq U_1$ and $U_1 \cap U_2 \subseteq U_2$; and
(5) If I is a piece of information, $I \in U_1$ and $U_1 \subseteq U_2$, then $I \in U_2$.

Information intervals are information sets, and thus, all the statements in Lemma 3.2 are inherited from set theory.

Next we will introduce the concept of neighborhood.

Definition 3.3 *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space, and let $I \in \mathcal{S}$. Let $[X, Y]$ be a non-void interval in the informalogical space. If $I \in [X, Y]$, which means $X \preceq I \preceq Y$, we say that the interval $[X, Y]$ is an $\mathcal{I}$-neighborhood, or neighborhood for short, of I, and we use $U_{(I)}[X, Y]$ to denote this relationship. We can simply use $[X, Y]$, $U_{(I)}$ or U to denote a neighborhood if no confusions seem possible.*

Definition 3.4 *We say that the family of all neighborhoods of a piece of information I is the neighborhood system of I, and we often use $\mathcal{U}_I$ to denote the neighborhood system of I.*

If $\mathcal{U}_0 \subseteq \mathcal{U}_I$, and every neighborhood of I contains a member of $\mathcal{U}_0$ as subinterval, then we say that $\mathcal{U}_0$ is a base for the neighborhood system of I, or a local base at I.

It is obvious that, in the informalogical space $(\mathcal{S}, \mathcal{I})$, the information interval $[0, \Omega]$ is a neighborhood of any piece of information in the space, where Ω is the reference information for the space. Thus, the neighborhood system of any piece of information is non-void.

Theorem 3.2 *The intersection of any finite number of neighborhoods of a piece of information is still a neighborhood of that piece of information.*

Proof Suppose $[X_1, Y_1], [X_2, Y_2], ..., [X_n, Y_n] \in \mathcal{U}_I$. Then, $X_i \preceq I \preceq Y_i$ for $i = 1, 2, ..., n$, and consequently, by 1 and 2 of Theorem 2.2, $\vee_{i=1}^n X_i \preceq I \preceq \wedge_{i=1}^n Y_i$. This implies that $\cap_{i=1}^n [X_i, Y_i] = [\vee_{i=1}^n X_i, \wedge_{i=1}^n Y_i]$ is really a neighborhood of I. $\square$

Using neighborhood, we can introduce the concept of accumulation information.

Definition 3.5 *Let $\mathcal{A}$ be an information set in the space. Let I be a piece of information in the space. We say that I is a piece of $\mathcal{I}$-accumulation informa-*

tion, or a piece of accumulation information for short, of the information set $\mathcal{A}$ if every neighborhood of I contains a member of $\mathcal{A}$ that is different from I itself.

4 Information Nets and Convergence

In this section, we will introduce the concept of information net. Information net is a generalization of information sequence. We will discuss the convergence of information nets in an informalogical space. The convergence of information nets is similar to the Moore-Smith convergence in general topology in that, for all major results of Moore-Smith convergence in general topology, we obtained similar results for the convergence of information net in an informalogical space. For Moore-Smith convergence in general topology, see [22, Chapter 2].

Like Moore-Smith convergence of nets in general topology, we need the concept of directed set. Directed set is a generalization of the natural numbers 1, 2, 3, ... with the natural order. The following definition comes from [22, Chapter 2].

Definition 4.1 *We say that a binary relation $\geq$ directs a set D, and that the pair $(D, \geq)$ is a directed set, if D is non-void, and*

(1) if m, n and p are members of D such that $m \geq n$ and $n \geq p$, then $m \geq p$;
(2) if $m \in D$, then $m \geq m$; and
(3) if m and n are members of D, then there is p in D such that $p \geq m$ and $p \geq n$.

Theorem 4.1 *Suppose I is a piece of information in the space, and $\mathcal{U}_I$ is its neighborhood system. Then, $(\mathcal{U}_I, \subseteq)$ is a directed set, where $\subseteq$ is the normal subset relation.*

Proof Notice that $[0, \Omega]$ is a neighborhood of I, where Ω is the reference information for the space. Thus, $\mathcal{U}_I$ is non-void.

Suppose $U_1, U_2, U_3 \in \mathcal{U}_I$ such that $U_1 \subseteq U_2$ and $U_2 \subseteq U_3$. Then, by Lemma 3.2, $U_1 \subseteq U_3$. For $U \in \mathcal{U}_I$, also by Lemma 3.2, $U \subseteq U$. Suppose that U_1 and U_2 are two members of $\mathcal{U}_I$. Let $V = U_1 \cap U_2$. Then, by Theorem 3.2, V is a member of $\mathcal{U}_I$, and by Lemma 3.2, $V \subseteq U_1$ and $V \subseteq U_2$.

Now, we know that $(\mathcal{U}_I, \subseteq)$ is really a directed set. □

To analyze the convergence of an information net, we also need the concept of product directed set of two directed sets. The following notions and Lemma 4.1 about product directed set all come from [22, Chapter 2].

Suppose $(D_1, \geq_1)$ and $(D_2, \geq_2)$ are two directed sets, and let $D = D_1 \times D_2$ be the Cartesian product of D_1 and D_2. We can construct a binary relation $\geq$ on D as the following: For $(m_1, n_1) \in D$ and $(m_2, n_2) \in D$, $(m_1, n_1) \geq (m_2, n_2)$ if and only if $m_1 \geq_1 m_2$ and $n_1 \geq_2 n_2$. Here, $m_1, m_2 \in D_1$ and $n_1, n_2 \in D_2$. The relation $\geq$ is called *product order* of $\geq_1$ and $\geq_2$, and the product order $\geq$ is also called *product relation* in this paper to indicate the fact that it is a "product" of two binary relations.

Lemma 4.1 *The product relation $\geq$ directs the Cartesian product $D = D_1 \times D_2$, and thus, $(D_1 \times D_2, \geq)$ is a directed set.*

$(D_1 \times D_2, \geq)$ is called the *product directed set* of $(D_1, \geq_1)$ and $(D_2, \geq_2)$.

With the above preparations completed, we will introduce the concepts of information net and its convergence in an informalogical space.

Definition 4.2 *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space, $(D, \geq)$ be a directed set and T be a function on D whose values are pieces of information in the space. That means, for each $n \in D$, there is one and only one $T_n \in \mathcal{S}$ that corresponds to n. Then, $\{T_n, n \in D, \geq\}$ is called an information net in the space $\mathcal{S}$. In case where no confusion seems possible, we simply use $\{T_n, n \in D\}$ or $\{T_n\}$ to denote an information net.*

Definition 4.3 *Let $\{T_n, n \in D, \geq\}$ be an information net, and $[X, Y]$ be an information interval. Then*

(1) we say that the information net $\{T_n, n \in D, \geq\}$ is in the information interval $[X, Y]$ if $T_n \in [X, Y]$ for every $n \in D$;
(2) we say that the information net $\{T_n, n \in D, \geq\}$ is eventually in the information interval $[X, Y]$ if there is $m \in D$ such that $T_n \in [X, Y]$ for every $n \in D$ that satisfies $n \geq m$; and
(3) we say that the information net $\{T_n, n \in D, \geq\}$ is frequently in the information interval $[X, Y]$ if for every $m \in D$ there is $n \in D$ such that $n \geq m$ and $T_n \in [X, Y]$.

Definition 4.4 *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space, $\{T_n, n \in D, \geq\}$ be an information net in the space, and I be a piece of information in the space. We say that the information net $\{T_n, n \in D, \geq\}$ converges to the information I in the informalogical space $(\mathcal{S}, \mathcal{I})$, or say that $\{T_n, n \in D, \geq\}$ $\mathcal{I}$-converges to I, if the information net $\{T_n, n \in D, \geq\}$ is eventually in every neighborhood of I. Information I is called a piece of $\mathcal{I}$-limit information of the information net $\{T_n, n \in D, \geq\}$ if $\{T_n, n \in D, \geq\}$ $\mathcal{I}$-converges to I. When no confusion seems possible, for short, we simply say that the information net $\{T_n, n \in D, \geq\}$*

converges to information I, and information I is a piece of limit information of the information net $\{T_n, n \in D, \geq\}$.

It is worthy to point out that an information net may converge to more than one piece of limit information: Let $\mathcal{S}$ be a non-void information set that contains 0 and $\Omega = \vee\mathcal{S}$. Let $\mathcal{I} = \{0, \Omega\}$ be the trivial informalogy for the space $\mathcal{S}$. Then, in this informalogical space $(\mathcal{S}, \mathcal{I})$, every information net converges to every piece of information in the the space $\mathcal{S}$. This simple example shows that, generally, there is no uniqueness of limit information. However, in a separated informalogical space we will introduce below, and only in a separated informalogical space, the limit information of every information net, if exists, is unique.

Definition 4.5 *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space. We say that $(\mathcal{S}, \mathcal{I})$ is a separated informalogical space, or say that it is separated, if for every two distinct pieces of information I and J in the space, i.e., $I, J \in \mathcal{S}$ and $I \neq J$, there exist neighborhoods U and V of I and J, respectively, such that $U \cap V = \theta$.*

Lemma 4.2 *Let U and V be two information intervals. Then, $U_1 \cap U_2 \neq \theta$ if and only if there is a piece of information $I \in \mathcal{S}$ such that $I \in U_1$ and $I \in U_2$.*

This lemma is obvious.

Theorem 4.2 *An informalogical space is separated if and only if every information net in the space has at most one piece of limit information.*

Proof Assume the informalogical space $(\mathcal{S}, \mathcal{I})$ is separated. Suppose $I, J \in \mathcal{S}$ and $I \neq J$. Then, there are neighborhoods U and V of I and J, respectively, such that $U \cap V = \theta$. By Lemma 4.2, an information net cannot be eventually in both U and V. Thus, an information net cannot converge to both I and J.

To establish the converse, assume $(\mathcal{S}, \mathcal{I})$ is not separated. We will construct an information net that converges to two distinct pieces of information. Since $(\mathcal{S}, \mathcal{I})$ is not separated, there exist two distinct pieces of information I and J in the space such that every neighborhood U of I intersects every neighborhood V of J. By Lemma 4.2, we can select a piece of information $T_{(U,V)} \in \mathcal{S}$ such that $T_{(U,V)} \in U$ and $T_{(U,V)} \in V$. Let $\mathcal{U}_I$ be the neighborhood system of I, and $\mathcal{U}_J$ be the neighborhood system of J. By Theorem 4.1, $(\mathcal{U}_I, \subseteq)$ and $(\mathcal{U}_J, \subseteq)$ are two directed sets. By Lemma 4.1, we can denote $(\mathcal{U}_I \times \mathcal{U}_J, \geq)$ as their product directed set. Then, $\{T_{(U,V)}, (U, V) \in \mathcal{U}_I \times \mathcal{U}_J, \geq\}$ is an information net in the space. Next, we will show that the information net $\{T_{(U,V)}, (U, V) \in \mathcal{U}_I \times \mathcal{U}_J, \geq\}$ converges to both I and J.

In fact, for every neighborhood U of I and every neighborhood V of J, we have $(U, V) \in \mathcal{U}_I \times \mathcal{U}_J$. For every $(U', V') \in \mathcal{U}_I \times \mathcal{U}_J$ that satisfies $(U', V') \geq (U, V)$, we have $U' \subseteq U$ and $V' \subseteq V$. We know $T_{(U',V')} \in U'$ and $T_{(U',V')} \in V'$ from the selection of $T_{(U',V')}$. Thus, $T_{(U',V')} \in U$ and $T_{(U',V')} \in V$. This means that the information net $\{T_{(U,V)}, (U,V) \in \mathcal{U}_I \times \mathcal{U}_J, \geq\}$ is eventually in every neighborhood U of I and every neighborhood V of J, and consequently, the information net converges to both I and J. $\square$

We introduced the concept of accumulation information of an information set at the end of Section 3. Our next theorem establishes the relationship between a piece of accumulation information of an information set and convergence of an information net.

Theorem 4.3 *A piece of information I is a piece of accumulation information of an information set $\mathcal{A}$ if and only if there exists an information net in $\mathcal{A} \setminus \{I\}$ that converges to I.*

Proof Assume I is a piece of accumulation information of $\mathcal{A}$. Let $\mathcal{U}_I$ be the neighborhood system of I. Then, for every $U \in \mathcal{U}_I$, U contains a member of $\mathcal{A}$ other than I itself. We denote this member of $\mathcal{A}$ as T_U. Then, it is clear that $T_U \in \mathcal{A} \setminus \{I\}$ and $T_U \in U$. By Theorem 4.1, $(\mathcal{U}_I, \subseteq)$ is a directed set. Thus, $\{T_U, U \in \mathcal{U}_I, \subseteq\}$ is an information net in $\mathcal{A} \setminus \{I\}$.

For every neighborhood U of I, namely $U \in \mathcal{U}_I$, when $V \in \mathcal{U}_I$ that satisfies $V \subseteq U$, we have $T_V \in U$ since $T_V \in V$ and $V \subseteq U$. This means that the information net $\{T_U, U \in \mathcal{U}_I, \subseteq\}$ is eventually in every neighborhood U of I, and consequently, this information net converges to I.

To establish the converse, assume that there exists an information net $\{T_n, n \in D, \geq\}$ in $\mathcal{A} \setminus \{I\}$ that converges to I. Then, for every neighborhood U of I, there is $p \in D$ such that $T_n \in U$ for every $n \in D$ that satisfies $n \geq p$. Since, by the definition of a directed set, $p \geq p$, we have $T_p \in U$. We know that $T_p \in \mathcal{A} \setminus \{I\}$. Thus, $T_p \neq I$. This means that every neighborhood of I contains a member of $\mathcal{A}$ other than I itself, and consequently, I is a piece of accumulation information of $\mathcal{A}$. $\square$

5 Information Subnets

First, we introduce the concepts of information subnet and cluster information. Then, we prove a theorem that establishes the relationship between cluster information and convergence of information subnets.

Definition 5.1 *Let $\{T_n, n \in D, \geq\}$ and $\{R_m, m \in E, \geq_1\}$ be two information nets. We say that $\{T_n, n \in D, \geq\}$ is an information subnet, or subnet for* *short, of* $\{R_m, m \in E, \geq_1\}$ *if there exists a function* N *on* D *with values in* E *such that*

(1) $T = R \bullet N$*, or equivalently,* $T_n = R_{N_n}$ *for each* $n \in D$*, where "*$\bullet$*" is the function composition; and*
(2) for each $m \in E$*, there is* $p \in D$ *such that if* $n \geq p$*, then,* $N_n \geq_1 m$*.*

Definition 5.2 *We say that a piece of information* I *is a piece of cluster information of an information net* $\{T_n\}$ *if the information net* $\{T_n\}$ *is frequently in every neighborhood of* I*.*

A piece of cluster information of an information net is different from a piece of accumulation information of an information set. To be a piece of accumulation information of an information set, every neighborhood of that piece of information should contain a member of the information set that is different from that piece of information itself. However, to be a piece of cluster information of an information net, that distinction is not required. For example, let $T_n \equiv I$ for each $n \in D$, where $(D, \geq)$ is a directed set and I is a piece of information. Then, I is a piece of cluster information of the information net $\{T_n, n \in D, \geq\}$. However, I is not a piece of accumulation information of the information set $\{T_n | n \in D\}$.

Lemma 5.1 *Suppose that* $\{R_m, m \in E, \geq_1\}$ *is an information net,* I *is a piece of information, and* $\mathcal{U}_I$ *is the neighborhood system of* I*. If the information net is frequently in every member of* $\mathcal{U}_I$*, then, there is an information subnet of* $\{R_m, m \in E, \geq_1\}$ *that converges to* I*.*

Proof Let $U \in \mathcal{U}_I$. Since $\{R_m, m \in E, \geq_1\}$ is frequently in U, for each $m_0 \in E$ there is $m \in E$ such that $m \geq_1 m_0$ and $R_m \in U$. Let $D = \{(m, U) | m \in E, U \in \mathcal{U}_I \text{ and } R_m \in U\}$. We define a binary relation $\geq$ on D as the following: $(m_2, U_2) \geq (m_1, U_1)$ if and only if $m_2 \geq_1 m_1$ and $U_2 \subseteq U_1$.

First, we show that $(D, \geq)$ is a directed set. It is obvious that D is non-void. If $(m_3, U_3) \geq (m_2, U_2)$ and $(m_2, U_2) \geq (m_1, U_1)$, then, we have $m_3 \geq_1 m_2$ and $m_2 \geq_1 m_1$, and $U_3 \subseteq U_2$ and $U_2 \subseteq U_1$. These imply $m_3 \geq_1 m_1$ and $U_3 \subseteq U_1$. Thus, $(m_3, U_3) \geq (m_1, U_1)$. It is clear that $(m, U) \geq (m, U)$, since $m \geq_1 m$ and $U \subseteq U$. For any two members (m_1, U_1) and (m_2, U_2) of D, there is $n' \in E$ such that $n' \geq_1 m_1$ and $n' \geq_1 m_2$ since $(E, \geq_1)$ is a directed set. There is $V = U_1 \cap U_2 \in \mathcal{U}_I$ such that $V \subseteq U_1$ and $V \subseteq U_2$. For $V \in \mathcal{U}_I$ and $n' \in E$, since $\{R_m, m \in E, \geq_1\}$ is frequently in V, there is $n \in E$ such that $n \geq_1 n'$ and $R_n \in V$. Then, we know 1) $(n, V) \in D$; 2) $(n, V) \geq (m_1, U_1)$ since $n \geq_1 n' \geq_1 m_1$ and $V \subseteq U_1$; and 3) $(n, V) \geq (m_2, U_2)$ since $n \geq_1 n' \geq_1 m_2$ and $V \subseteq U_2$. Now we know that $(D, \geq)$ is a directed set.

Second, we construct a subnet of $\{R_m, m \in E, \geq_1\}$ as the following: For each $(m, U) \in D$, let $N_{(m,U)} = m$ and $T_{(m,U)} = R_m$. Then, $T = R \bullet N$. For each $n \in E$ and the neighborhood $U_0 \equiv [0, \Omega]$ of I, since the net $\{R_m, m \in E, \geq_1\}$ is frequently in U_0, there is $p \in E$ such that $p \geq_1 n$ and $R_p \in U_0$. Then, $(p, U_0) \in D$. For any $(m, U) \in D$ that satisfies $(m, U) \geq (p, U_0)$, we have $N_{(m,U)} = m \geq_1 p \geq_1 n$. Now we know that $\{T_{(m,U)}, (m, U) \in D, \geq\}$ is a really subnet of $\{R_m, m \in E, \geq_1\}$.

Finally, we show that the net $\{T_{(m,U)}, (m, U) \in D, \geq\}$ converges to I. For every neighborhood $V \in \mathcal{U}_I$, since $\{R_m, m \in E, \geq_1\}$ is frequently in V, there is $p \in E$ such that $R_p \in V$. Thus, $(p, V) \in D$. If $(m, U) \in D$ and $(m, U) \geq (p, V)$, then $T_{(m,U)} = R_m \in U \subseteq V$, and consequently, $T_{(m,U)} \in V$. This means that the information net $\{T_{(m,U)}, (m, U) \in D, \geq\}$ is eventually in every neighborhood V of I. Thus, the net $\{T_{(m,U)}, (m, U) \in D, \geq\}$ converges to I. $\square$

Theorem 5.1 *A piece of information I is a piece of cluster information of an information net $\{R_m, m \in E, \geq_1\}$ if and only if the information net has a subnet that converges to I.*

Proof Suppose that I is a piece of cluster information of $\{R_m, m \in E, \geq_1\}$. Then, $\{R_m, m \in E, \geq_1\}$ is frequently in every neighborhood of I. Thus, by Lemma 5.1, the information net has a subnet that converges to I.

To establish the converse, suppose that $\{R_m, m \in E, \geq_1\}$ has a subnet $\{T_n, n \in D, \geq\}$ that converges to I. Then, for every neighborhood U of I, there is $p_1 \in D$ such that if $n \in D$ and $n \geq p_1$, then $T_n \in U$. For every $m \in E$, since $\{T_n, n \in D, \geq\}$ is a subnet of $\{R_m, m \in E, \geq_1\}$, there is $p_2 \in D$ such that if $n \in D$ and $n \geq p_2$, then $N_n \geq_1 m$.

Let's go back to D. Since $(D, \geq)$ is a directed set, for $p_1 \in D$ and $p_2 \in D$, there is $q \in D$ such that $q \geq p_1$ and $q \geq p_2$. Then, we have $T_q \in U$ since $q \geq p_1$, and we have $N_q \geq_1 m$ since $q \geq p_2$. Let $m' = N_q$. Then, we have $m' \in E$, $m' \geq_1 m$ and $R_{m'} = R_{N_q} = T_q \in U$. That means, for every neighborhood U of I and every $m \in E$, there is $m' \in E$ such that $m' \geq_1 m$ and $R_{m'} \in U$. Thus, $\{R_m, m \in E, \geq_1\}$ is frequently U. Consequently, I is a piece of cluster information of the information net $\{R_m, m \in E, \geq_1\}$. $\square$

6 Information Sequences

In this section, we will discuss a special type of information nets, information sequence.

Definition 6.1 *We say that an information net* $\{T_n, n \in D, \geq\}$ *is an information sequence if there is a bijective map between* D *and the set of positive integers* $\{1, 2, 3, ...\}$ *that preserves the order. That is, suppose* f *is the bijective map from* D *to* $\{1, 2, 3, ...\}$*, then for any* $n_1, n_2 \in D$*,* $n_1 \geq n_2$ *in* D *if and only if* $f(n_1) \geq f(n_2)$ *in* $\{1, 2, 3, ...\}$.

Without loss of generality, in what follows, we assume that $D = \{1, 2, ..., n, ...\}$, and $\geq$ is the usual order of positive integers. We can write an information sequence as $\{T_1, T_2, ..., T_n, ...\}$. For simplicity, we often simply write $\{T_1, T_2, ..., T_n, ...\}$ as $\{T_n\}$.

Definition 6.2 *Let* $\{T_n\}$ *and* $\{R_m\}$ *be two information sequences. We say that* $\{T_n\}$ *is an information subsequence, or subsequence for short, of* $\{R_m\}$*, if, viewed as two information nets,* $\{T_n\}$ *is a subnet of* $\{R_m\}$.

If $\{T_n\}$ is an information subsequence of $\{R_m\}$, then there is a function N on positive integers and values in positive integers such that $T_i = R_{N_i}$ for $i = 1, 2, ..., n, ...$, and for each positive integer m, there is positive integer n such that if $i \geq n$, then $N_i \geq m$.

In Section 3, we have introduced the concept of base of the neighborhood family of a piece of information I, or in other words, the local base at I. Now, we use that concept to define a particular type of informalogical spaces, first countable informalogical spaces. A first countable informalogical space has good properties for information sequences.

Definition 6.3 *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space. We say that the informalogical space is first countable if the neighborhood family of each piece of information in the space has a countable base. In other words, there is a countable local base at each piece of information in the space.*

A set is *countable* means that there exists a bijective map from that set to the set of positive integers $\{1, 2, 3, ...\}$. For an example, the set of all integers is countable. For another example, the set of all real numbers is uncountable, since it is impossible to establish any bijective map between the set of all real numbers and the set of positive integers. Also, the set of all subsets of the set of positive integers is uncountable. For information about computability theory, see [10,11,43].

Theorem 6.1 *Suppose that the informalogical space* $(\mathcal{S}, \mathcal{I})$ *is first countable,* I *is a piece of information in the space,* $\mathcal{A}$ *is an information set in the space, and* $\{R_m\}$ *is an information sequence in the space. Then*

(1) I *is a piece of accumulation information of* $\mathcal{A}$ *if and only if there is an information sequence in* $\mathcal{A} \setminus \{I\}$ *that converges to* I*; and*
(2) I *is a piece of cluster information of* $\{R_m\}$ *if and only if* $\{R_m\}$ *has a*

subsequence that converges to I.

Proof Since $(\mathcal{S}, \mathcal{I})$ is first countable, we can assume that $\{U_1, U_2, ..., U_n, ...\}$ is a local base at I. Then, for $n = 1, 2, 3, ...$, we define $V_n \equiv \cap_{i=1}^{n} U_i$. By Theorem 3.2, $V_1, V_2, ..., V_n, ...$ are also neighborhoods of I. It is obvious that $... \subseteq V_{n+1} \subseteq V_n \subseteq ... \subseteq V_2 \subseteq V_1$. Also, $V_n \subseteq U_n$ for $n = 1, 2, 3, ...$ Thus, $\{V_1, V_2, ..., V_n, ...\}$ is also a local base at I.

(1) Suppose I is a piece of accumulation information of $\mathcal{A}$. Then, for each V_n, there is $T_n \in \mathcal{A} \setminus \{I\}$ such that $T_n \in V_n$, and consequently, we obtain an information sequence $\{T_1, T_2, ..., T_n, ...\}$ in $\mathcal{A} \setminus \{I\}$. For every neighborhood U of I, since $\{V_1, V_2, ..., V_n, ...\}$ is a local base at I, there is some V_p such that $V_p \subseteq U$. Then, when $n \geq p$, we have $T_n \in V_n \subseteq V_p \subseteq U$, and thus $T_n \in U$. This means that the information sequence $\{T_1, T_2, ..., T_n, ...\}$ is eventually in every neighborhood of I. Therefore, $\{T_1, T_2, ..., T_n, ...\}$ converges to I.

Since an information sequence is also an information net, the converse part is established by Theorem 4.3.

(2) Suppose I is a piece of cluster information of the information sequence $\{R_m\}$. Then, for each V_i, there is a positive integer N_i such that $N_i \geq i$ and $R_{N_i} \in V_i$. Let $T_i = R_{N_i}$ for $i = 1, 2, 3, ...$ It is clear that $\{T_n\}$ is a subsequence of $\{R_m\}$. For every neighborhood U of I, since $\{V_1, V_2, ..., V_n, ...\}$ is a local base at I, there is some V_p such that $V_p \subseteq U$. Then, when $n \geq p$, we have $T_n = R_{N_n} \in V_n \subseteq V_p \subseteq U$, and thus $T_n \in U$. This means that the information sequence $\{T_1, T_2, ..., T_n, ...\}$ is eventually in every neighborhood of I. Therefore, $\{T_1, T_2, ..., T_n, ...\}$ converges to I.

Since an information subsequence is also an information subnet, the converse part is established by Theorem 5.1.

□

7 Future Work

One of our future works will be to continue the theoretical build up in the informalogical space. For example, we will discuss difference of information and decompositions of information. As we have mentioned in Section 2, some of the theoretical build up have mirror concepts in general topology in a symbolic sense, and others do not.

As already pointed out, the current work does not take fuzziness into consideration. This is a limitation of the current work. Future works should take fuzziness into consideration to generate more general approaches.

Application is also an important area of future works.

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