Informalogical Space: Open Information Intervals, Open Convergence and Open Compactness

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Abstract

In previous works, we presented a meaning based information theory in which the core concept is an *informalogical space*. We did various discussions in informalogical spaces. We introduced *information nets* which is a generalization of information sequences. We introduced *information intervals*, *neighborhoods* of a piece of information, and *convergence* of information nets. We built a Moore-Smith style convergence theory in informalogical spaces.

In this paper, we show an undesirable property of convergence of information nets in the previous works. Then, we introduce *open information intervals*, *open neighborhoods* of a piece of information, and *open convergence* of information nets. Open convergence avoids the undesirable property of convergence in the previous works. However, at the same time, we point out a limitation of open convergence: for open convergence, the Moore-Smith style convergence theory cannot be established in general informalogical spaces. The Moore-Smith style convergence theory can only be established in informalogical spaces that satisfy certain conditions.

In this paper, we also introduce *open compactness* of informalogical spaces and prove that an informalogical space is open compact if and only if each information net in the informalogical space has a subnet that open converges.

Key words: informalogical space, open information interval, open neighborhood, information net, open convergence, open compact.

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1. Introduction

In [1], we presented a meaning based information theory, a theory of information that is based on meanings of information and relationships among information. We introduced the core concept of our approach, *informalogical space*. We introduced the concept of an *information interval* in an informalogical space and, based on information intervals, we introduced the concept of a *neighborhood* of a piece of information in an informalogical space. We introduced the concept of an *information net* in an informalogical space. Based on the concept of neighborhood, we built a *convergence* theory for information nets in an informalogical space. The convergence theory is similar to the Moore-Smith convergence theory in general topology in that, for all major results of Moore-Smith convergence theory (see [3, Chapter 2]), we obtained similar results for the convergence of information nets in an informalogical space.

In [2], based on the concept of information interval, we introduced the concept of *interval cover* of an information set in an informalogical space and, based on the concept of interval cover, we introduced the concept of a *compact* informalogical space. We proved that an informalogical space is compact if and only if each information net in the informalogical space has a subnet that converges.

In this paper, we show an undesirable property of the convergence of information nets that was introduced in [1]. We also introduce a new type of information intervals, and thus introduce a new type of neighborhoods of a piece of information in an informalogical space. Based on the new type of neighborhoods, we introduce a new type of convergence of information nets in an informalogical space. The new type of convergence avoids the undesirable property and under certain conditions the new type of convergence preserves all results of the convergence in [1].

Also, based on the new type of information intervals, we introduce a new type of interval covers of an information set in an informalogical space and thus introduce a new type of compactness of informalogical spaces. For the new type of compactness and convergence, we prove that an informalogical space is compact if and only if each information net in the informalogical space has a subnet that converges.

Before showing an undesirable property of the convergence of information nets in [1] and introducing a new type of convergence for information nets, we revisit some relevant concepts that were introduced in [1].

The *contain* relation between two pieces of information, and the *union* and *intersection* operations on information were introduced in [1]. Basically, suppose I and J are two pieces of information. If I can be inferred from J, then information I is *contained* in information J, and we can also say that information J *contains* information I. This relation is represented as $I \preceq J$ or $J \succeq I$.

As for the union and intersection operations on a non-empty information set $\mathcal{A}$, basically, the union $\vee\mathcal{A}$ is the sum of all the pieces of information in the information set $\mathcal{A}$, and the intersection $\wedge\mathcal{A}$ is the common information that is contained in each piece of information in the information set $\mathcal{A}$.

Below is the core concept of our approach, *informalogical space*, which was introduced in [1].

Definition 1.1. ([1]) *Let $\mathcal{S}$ be a non-empty information set, and let $\Omega = \vee\mathcal{S}$ (i.e., Ω is the union of all the information in $\mathcal{S}$). Let $\mathcal{I}$ be a non-empty subset of $\mathcal{S}$ such that $\vee\mathcal{I} = \Omega$. We say that $\mathcal{I}$ is an informalogy, that $\mathcal{S}$ is the space of the informalogy $\mathcal{I}$, that $\mathcal{I}$ is an informalogy for the space $\mathcal{S}$ and that the pair $(\mathcal{S}, \mathcal{I})$ is an informalogical space, if the following two conditions hold:*

1. *if $I, J \in \mathcal{I}$, then $I \wedge J \in \mathcal{I}$; and*
2. *if $\mathcal{I}_0 \subseteq \mathcal{I}$, then $\vee\mathcal{I}_0 \in \mathcal{I}$.*

After revisit of the above basic concepts, we revisit two theorems in [2] that contain some basic properties of information and information sets. As the same as assumed in [1] and [2], in this paper, all pieces of information under discussion are consistent information.

Theorem 1.1. ([2]) *Suppose that A, B, X and Y are pieces of information, and 0 is the zero information. Then,*

1. *$A \preceq A$;*
2. *if $X \preceq A$ and $A \preceq Y$, then $X \preceq Y$;*
3. *if $A \preceq X$ and $B \preceq X$, then $A \vee B \preceq X$;*
4. *if $X \preceq A$ and $X \preceq B$, then $X \preceq A \wedge B$;*
5. *$A \wedge B \preceq A \vee B$;*
6. *$A \vee B = B \vee A$, $A \wedge B = B \wedge A$;*
7. *if $X \preceq A$ and $Y \preceq B$, then $X \vee Y \preceq A \vee B$ and $X \wedge Y \preceq A \wedge B$;*
8. *$X \vee (A \vee B) = (X \vee A) \vee B$, $X \wedge (A \wedge B) = (X \wedge A) \wedge B$; and*

9. $A \preceq B$, $A \vee B = B$ and $A \wedge B = A$ are equivalent.

Theorem 1.2. ([2]) *Suppose that X, A and B are pieces of information, and $\mathcal{A}$ and $\mathcal{B}$ are information sets. Then,*

1. *if $A \preceq X$ for each $A \in \mathcal{A}$, then $\vee\mathcal{A} \preceq X$; if $X \preceq A$ for each $A \in \mathcal{A}$, then $X \preceq \wedge\mathcal{A}$;*
2. *if for each $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $A \preceq B$, then $\vee\mathcal{A} \preceq \vee\mathcal{B}$;*
3. *$X \vee (\vee\mathcal{A}) = \vee\{X \vee A | A \in \mathcal{A}\}$, $X \wedge (\wedge\mathcal{A}) = \wedge\{X \wedge A | A \in \mathcal{A}\}$; and*
4. *$(\vee\mathcal{A}) \vee (\vee\mathcal{B}) = \vee\{A \vee B | A \in \mathcal{A}, B \in \mathcal{B}\}$, $(\wedge\mathcal{A}) \wedge (\wedge\mathcal{B}) = \wedge\{A \wedge B | A \in \mathcal{A}, B \in \mathcal{B}\}$.*

Next, we revisit the concepts of *information interval*, or *interval* for short, and *neighborhood* introduced in [1].

Definition 1.2. ([1]) *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space. Let X and Y be two members of the informalogy $\mathcal{I}$. We define $[X, Y]$ as $[X, Y] \equiv \{I | I \in \mathcal{S}$ and $X \preceq I \preceq Y\}$. $[X, Y]$ is an information set which contains all the information in $\mathcal{S}$ that ranges from the lower endpoint X to the upper endpoint Y. We call $[X, Y]$ an information interval in the informalogical space $(\mathcal{S}, \mathcal{I})$, or simply an interval. When $[X, Y]$ is non-empty, we call it a non-empty interval; when $[X, Y]$ is empty, we call it an empty interval, and we use θ to denote an empty interval.*

When both $[X_1, Y_1]$ and $[X_2, Y_2]$ are information intervals, and $[X_1, Y_1] \subseteq [X_2, Y_2]$, we say that $[X_1, Y_1]$ is a subinterval of $[X_2, Y_2]$.

A set of intervals is called a family of intervals, or an interval family. We often use $\mathcal{U}$ to represent an interval family.

In cases where no confusion is likely to result, we may simply use a single letter such as U, V, etc. to represent an information interval. However, it should be kept in mind that an information interval is not a single piece of information, but a set of information.

Definition 1.3. ([1]) *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space, and let $I \in \mathcal{S}$. Let $[X, Y]$ be a non-empty interval in the informalogical space. If $I \in [X, Y]$, which means $X \preceq I \preceq Y$, we say that the interval $[X, Y]$ is an $\mathcal{I}$-neighborhood, or neighborhood for short, of I, and we use $U_{(I)}[X, Y]$ to denote this relationship. We can simply use $[X, Y]$, $U_{(I)}$ or U to denote a neighborhood if no confusion seems possible.*

Next, we revisit the concept of *information net* in an informalogical space, and the concept of *convergence* of information nets in an informalogical space that were introduced in [1].

Similar to the concept of nets in general topology ([3, Chapter 2]), for presenting the concept of an information net, we need the concept of a directed set. A directed set is a generalization of the natural numbers 1, 2, 3, ... with the natural order. The following definition comes from [3, Chapter 2].

Definition 1.4. ([3, Chapter 2]) *We say that a binary relation $\geq$ directs a set D, and that the pair $(D, \geq)$ is a directed set, if D is non-empty, and*

1. *if m, n and p are members of D such that $m \geq n$ and $n \geq p$, then $m \geq p$;*
2. *if $m \in D$, then $m \geq m$; and*
3. *if m and n are members of D, then there is p in D such that $p \geq m$ and $p \geq n$.*

Below is the definition of an information net in an informalogical space.

Definition 1.5. ([1]) *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space, $(D, \geq)$ be a directed set and T be a function on D whose values are pieces of information in the space. That means, for each $n \in D$, there is one and only one $T_n \in \mathcal{S}$ that corresponds to n. Then, $\{T_n, n \in D, \geq\}$ is called an information net in the space $\mathcal{S}$. In cases where no confusion would result, we simply use $\{T_n, n \in D\}$ or $\{T_n\}$ to denote an information net.*

Next, we revisit the concept of *convergence* of information nets in an informalogical space that was introduced in [1].

Definition 1.6. ([1]) *Let $\{T_n, n \in D, \geq\}$ be an information net, and let $[X, Y]$ be an information interval. Then*

1. *we say that the information net $\{T_n, n \in D, \geq\}$ is in the information interval $[X, Y]$ if $T_n \in [X, Y]$ for every $n \in D$;*
2. *we say that the information net $\{T_n, n \in D, \geq\}$ is eventually in the information interval $[X, Y]$ if there is an $m \in D$ such that $T_n \in [X, Y]$ for every $n \in D$ that satisfies $n \geq m$; and*

3. *we say that the information net* $\{T_n, n \in D, \geq\}$ *is frequently in the information interval* $[X, Y]$ *if, for every* $m \in D$*, there is* $n \in D$ *such that* $n \geq m$ *and* $T_n \in [X, Y]$*.*

Definition 1.7. ([1]) *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space,* $\{T_n, n \in D, \geq\}$ *be an information net in the space, and* I *be a piece of information in the space. We say that the information net* $\{T_n, n \in D, \geq\}$ *converges to the information* I *in the informalogical space* $(\mathcal{S}, \mathcal{I})$*, or say that* $\{T_n, n \in D, \geq\}$ $\mathcal{I}$*-converges to* I*, if the information net* $\{T_n, n \in D, \geq\}$ *is eventually in every neighborhood of* I*. The information* I *is called a piece of* $\mathcal{I}$*-limit information of the information net* $\{T_n, n \in D, \geq\}$ *if* $\{T_n, n \in D, \geq\}$ $\mathcal{I}$*-converges to* I*. When no confusion would arise, for short, we simply say that the information net* $\{T_n, n \in D, \geq\}$ *converges to information* I*, and that the information* I *is a piece of limit information of the information net* $\{T_n, n \in D, \geq\}$*.*

With the introduction of information nets and convergence of an information net, we established in [1] a convergence theory of information nets in an informalogical space. That convergence theory is similar to the Moore-Smith convergence theory in a topological space in that, for all major results of Moore-Smith convergence in a topological space (see [3, Chapter 2]), we obtained similar results for the convergence of information nets in an informalogical space.

Now, we show an undesirable property of the convergence of information nets in Definition 1.7: for an information net $\{T_n, n \in D, \geq\}$ to converge to a piece of information I that is in the informalogy $\mathcal{I}$ (*i.e.*, $I \in \mathcal{I}$), it is necessary that there is an $m \in D$ such that $T_n = I$ for every $n \in D$ that satisfies $n \geq m$, or in other words, T_n needs to be eventually identical to I. This is shown in the following theorem.

Theorem 1.3. *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space,* $\{T_n, n \in D, \geq\}$ *be an information net in the informalogical space, and* $I \in \mathcal{I}$*. Then,* $\{T_n, n \in D, \geq\}$ *converges to* I *if and only if there is an* $m \in D$ *such that* $T_n = I$ *for every* $n \in D$ *that satisfies* $n \geq m$*.*

Proof. It is obvious from the definitions that $\{T_n, n \in D, \geq\}$ converges to I if there is an $m \in D$ such that $T_n = I$ for every $n \in D$ that satisfies $n \geq m$. On the other hand, suppose that $\{T_n, n \in D, \geq\}$ converges to I. Since $I \in \mathcal{I}$, $[I, I]$ is an information interval. Actually, $[I, I]$ is a neighborhood of I since

$I \in [I, I]$. Thus, by Definition 1.6 and Definition 1.7, there is an $m \in D$ such that $T_n \in [I, I]$ for every $n \in D$ that satisfies $n \geq m$. $T_n \in [I, I]$ means $I \preceq T_n \preceq I$, and thus, $T_n = I$. □

The property that, when I is a member of the informalogy $\mathcal{I}$ (*i.e.*, $I \in \mathcal{I}$), T_n needs to be eventually identical to I for the information net $\{T_n, n \in D, \geq\}$ to converge to I is undesirable, since the requirement for the convergence is too strong. However, it also should be pointed out that this undesirable property only exists if the information net were to converge to a member of the informalogy. There is no such an undesirable property for the information net to converge to a piece of information which is not a member of the informalogy.

The above undesirable property is a consequence of the definition of information interval. The information interval $[X, Y]$ in Definition 1.2 is a "closed" interval in that the information interval contains the two end points X and Y. Then, when $I \in \mathcal{I}$, $[I, I]$ is an information interval and thus is a neighborhood of I. The fact that $[I, I]$ only contains I results in that T_n needs to be eventually identical to I for the information net $\{T_n, n \in D, \geq\}$ to converge to I.

In this paper, we introduce a different type of information intervals, *open information intervals*. Based on open information intervals, we introduce *open neighborhoods* of a piece of information, and based on open neighborhoods, we introduce a new type of convergence of information nets, *open convergence*. The open convergence avoids the undesirable property discussed above. However, we also point out the limitation of open convergence comparing with the *closed convergence* introduced in [1] and revisited in Definition 1.7: In [1], the Moore-Smith style convergence theory for closed convergence was established for general case of informalogical spaces, but for open convergence, a same convergence theory cannot be established for general case of informalogical spaces. For open convergence, a same convergence theory can only be established for informalogical spaces that satisfy certain conditions.

In Section 2, we introduce the concept of an *open information interval*, and based on that concept, we introduce the concept of an *open neighborhood* of a piece of information. Unlike the information intervals introduced in [1] and revisited in Definition 1.2, the intersection of two open information intervals is not necessarily an open information interval. We give an informalogical space satisfying the condition that the intersection of two open information intervals is still an open information interval a special name: *open normal informalogical space*.

In Section 3, we introduce the concept of *open convergence* of information nets based on open neighborhoods. Open convergence avoids the undesirable property of closed convergence shown in Theorem 1.3. However, for open convergence, we cannot establish a Moore-Smith style convergence theory in general informalogical spaces as we did for closed convergence in [1]. For open convergence, we can only establish the Moore-Smith style convergence theory in open normal informalogical spaces.

In Section 4, we discuss open convergence of information subnets and information sequences in open normal informalogical spaces.

In Section 5, based on the concept of open interval, we introduce the concept of *open interval cover* of an information set in an informalogical space, and based on the concept of open interval cover, we introduce the concept of an *open compact* informalogical space. We prove 1) an informalogical space $(\mathcal{S}, \mathcal{I})$ is open compact if and only if each information net in the informalogical space has a piece of open cluster information, and 2) if $(\mathcal{S}, \mathcal{I})$ further is an open normal informalogical space, then $(\mathcal{S}, \mathcal{I})$ is open compact if and only if each information net in the informalogical space has a subnet that open converges.

In Section 6, we discuss various properties of open intervals and open neighborhoods under isomorphisms introduced in [2]. We also prove that open limit uniqueness, open normality, open separatedness, open first countability and open compactness are all isomorphic invariants. An isomorphic invariant is a property of an informalogical space that is preserved under isomorphisms (see [2]).

In Section 7, we conclude and briefly discuss some of our future work.

2. Open Information Intervals and Open Neighborhoods

In this section, we introduce the concept of an *open information interval*, and based on that concept, we introduce the concept of an *open neighborhood* of a piece of information. We discuss differences between open information intervals and the information intervals introduced in [1], or in other words, *closed information intervals*. We also introduce a special type of informalogical spaces, *open normal informalogical spaces*. In open normal informalogical spaces, for *open convergence* (to be introduced in next section), we can establish a Moore-Smith style convergence theory for information nets.

Before introducing the concept of an open information interval, we revisit a concept we introduced in [1], the *reference information* Ω. For an informalogical space $(\mathcal{S}, \mathcal{I})$, $\Omega = \vee\mathcal{S}$. Also, $\Omega = \vee\mathcal{I}$ since $\vee\mathcal{S} = \vee\mathcal{I}$. (See [1].) We know that any informalogy $\mathcal{I}$ contains at least two members, the zero information 0 and the reference information Ω. We call a piece of information in $(\mathcal{S}, \mathcal{I})$ that is neither zero information 0 nor reference information Ω a piece of *proper information* in the informalogical space $(\mathcal{S}, \mathcal{I})$. Also, if $I \prec J$ (*i.e.*, $I \preceq J$ and $I \neq J$), we say that I is a *proper subinformation* of J.

Definition 2.1. *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space. Let X and Y be two members of the informalogy $\mathcal{I}$. We define (X, Y) as $(X, Y) \equiv \{I | I \in \mathcal{S}$ and $X \prec I \prec Y\}$. (X, Y) is an information set which contains all the information in $\mathcal{S}$ that ranges from the lower endpoint X to the upper endpoint Y and that is not X or Y. We call (X, Y) a proper open information interval in the informalogical space $(\mathcal{S}, \mathcal{I})$, or simply a proper open interval.*

We define $[0, Y)$ as $[0, Y) \equiv \{I | I \in \mathcal{S}$ and $0 \preceq I \prec Y\}$, and we define $(X, \Omega]$ as $(X, \Omega] \equiv \{I | I \in \mathcal{S}$ and $X \prec I \preceq \Omega\}$. We call $[0, Y)$ a lower special open information interval in the informalogical space $(\mathcal{S}, \mathcal{I})$, or simply a lower special open interval, and we call $(X, \Omega]$ an upper special open information interval in the informalogical space $(\mathcal{S}, \mathcal{I})$, or simply an upper special open interval.

The aggregation of proper open information intervals and special open information intervals are called open information intervals, or simply open intervals. Thus, an open interval can be either a proper open interval or a special open interval. We can use a single letter, such as U, to represent an open interval. However, it should be kept in mind that an open interval is an information set, not a single piece of information.

When both U and V are open intervals (proper open intervals), and $U \subseteq V$, we say that U is an open subinterval (proper open subinterval) of V.

A set of open (proper open) intervals is called a family of open (proper open) intervals, or an open (proper open) interval family.

The reason why we include special open intervals (*i.e.*, lower and upper special open intervals) in open intervals in addition to proper open intervals is that otherwise there would be no open intervals that can contain the two special (and also trivial) pieces of information 0 and Ω, and consequently, 0 and Ω would not be able to have open neighborhoods (to be introduced later). Then, it would be theoretically impossible for any information net to converge to either 0 or Ω. That would look conceptually bad even though it may not materially bad. The inclusion of special open intervals at least avoids that conceptual drawback. However, since proper open intervals are the natural and important open intervals, when we later introduce concepts and establish theorems based on open intervals, we also introduce parallel concepts and establish parallel theorems based on proper open intervals.

With the above definition of open information interval introduced, we can call the intervals introduced in [1] (see Definition 1.2 above) *closed information intervals*.

There are some differences between closed information intervals and open information intervals. One difference is: the expression of a closed interval is unique in that if $[X_1, Y_1]$ and $[X_2, Y_2]$ are two non-empty closed intervals, then, $[X_1, Y_1] = [X_2, Y_2]$ if and only if $X_1 = X_2$ and $Y_1 = Y_2$ (see Lemma 3.1 of [1]), but on the other hand, the expression of an open interval is not generally unique.

A more material difference between closed information intervals and open information intervals is: if $[X_1, Y_1]$ and $[X_2, Y_2]$ are two closed intervals, then, as two information sets, their intersection set $[X_1, Y_1] \cap [X_2, Y_2]$ is also a closed interval (See Theorem 3.1 of [1]), but on the other hand, the intersection of two open intervals is not generally an open interval.

The following example exhibits the two characteristics of open information intervals: the expression of an open interval is not generally unique; and the intersection of two open intervals is not generally an open interval.

Assume a, b, c, d, e are independent pieces of information. For examples,

$a =$ "In 2011, New York City had a population of app proximately 8,244,910 people";

$b =$ "Texas has 254 counties";

$c =$ "Louisiana has a humid subtropical climate";

$d =$ "John is 38 years old"; and

$e =$ "Michael is a computational scientist".

We use these five independent pieces of information as building blocks to construct an informalogical space $(\mathcal{S}, \mathcal{I})$. We explain some notations first. We use $< a, b, c >$ to represent the union of a, b and c. That is, $< a, b, c >$ $= a \vee b \vee c$. For the examples of a, b, c mentioned above, $< a, b, c > =$ "In 2011, New York City had a population of approximately 8,244,910 people; Texas has 254 counties; and Louisiana has a humid subtropical climate." We use dORe to represent "Either d or e", which is also $d \wedge e$, the intersection of d and e. For the examples of d, e mentioned above, dOR$e =$ "Either John is 38 years old or Michael is a computational scientist."

With the above notations explained we can construct pieces of information that constitute the informalogical space $(\mathcal{S}, \mathcal{I})$:

$X_1 = < a >$.

$X_2 = < b >$.

$W = < a, b >$.

$I = < a, b, c >$.

$J = < a, b, d >$.

$K = < a, b, e >$.

$Z = < a, b, c, d\text{OR}e >$.

$Y_1 = < a, b, c, d >$.

$Y_2 = < a, b, c, e >$.

Note that the right side actually shows a *decomposition*. For an example, $\{a, b, c\}$ is a decomposition of I. (See [2] for definition of decomposition.)

It is easy to see that $W = X_1 \vee X_2$ and $Z = Y_1 \wedge Y_2$. Also, we have

$X_1 \prec W$, $X_2 \prec W$,

$W \prec I$, $W \prec J$, $W \prec K$,

$I \prec Z$,

$J \prec Y_1$, $Z \prec Y_1$,

$K \prec Y_2$, $Z \prec Y_2$.

Let

$\mathcal{S} = \{0, X_1, X_2, W, I, J, K, Z, Y_1, Y_2, \Omega\}$ and

$\mathcal{I} = \{0, X_1, X_2, W, Z, Y_1, Y_2, \Omega\}$

Then, $(\mathcal{S}, \mathcal{I})$ is an informalogical space. In this informalogical space,

$(X_1, Y_1) = \{W, I, J, Z\}$, and

$(X_2, Y_1) = \{W, I, J, Z\}$.

$(X_1, Y_1) = (X_2, Y_1)$ but $X_1 \neq X_2$. Similarly,

$(X_1, Y_2) = \{W, I, K, Z\}$, and

$(X_2, Y_2) = \{W, I, K, Z\}$.

$(X_1, Y_2) = (X_2, Y_2)$ but $X_1 \neq X_2$. This shows that the expression of an open interval is not generally unique.

In this informalogical space,

$(X_1, Y_1) = \{W, I, J, Z\}$,

$(X_2, Y_2) = \{W, I, K, Z\}$, and

$(X_1, Y_1) \cap (X_2, Y_2) = \{W, I, Z\}$.

Obviously $\{W, I, Z\}$ cannot be a special open interval since it contains neither 0 nor Ω. With respect to the possibility of $\{W, I, Z\}$ being a proper open interval, since $W, Z \in \{W, I, Z\}$, none of the end points of the proper open interval can be either W or Z. Then, the only proper open intervals left are (X_1, Y_1), (X_1, Y_2), (X_2, Y_1) and (X_2, Y_2). It is clear that none of those four proper open intervals equal to $\{W, I, Z\}$. Thus, the intersection $(X_1, Y_1) \cap (X_2, Y_2)$ of (X_1, Y_1) and (X_2, Y_2) cannot be an open interval. This shows that the intersection of two open intervals is not generally an open interval.

In the above example to show that the expression of an open interval is not generally unique, the two expressions of a same open interval has one common endpoint and one different endpoint. Actually, it is easy to modify the above example so that both endpoints of two expressions of a same interval are different. The modifications are taking out I, J, K, making $W = Z$ and making some other minor changes:

$X_1 = < a, d\text{OR}e >$.

$X_2 = < b, d\text{OR}e >$.

$W = < a, b, d\text{OR}e >$.

$Y_1 = < a, b, d >$.

$Y_2 = < a, b, e >$.

Then, it is easy to see that $W = X_1 \vee X_2 = Y_1 \wedge Y_2$. Let

$\mathcal{I} = \mathcal{S} = \{0, X_1, X_2, W, Y_1, Y_2, \Omega\}$.

Then, $(\mathcal{S}, \mathcal{I})$ is an informalogical space. In this informalogical space,

$(X_1, Y_1) = (X_2, Y_2) = \{W\}$, but $X_1 \neq X_2$ and $Y_1 \neq Y_2$.

As mentioned earlier, we sometimes simply use a single letter, such as U to denote an open interval. Since open intervals are information sets, the statements in the following lemma are inherited directly from set theory.

Lemma 2.1. *Suppose U, U_1, U_2 and U_3 are open information intervals. Then*

1. $U \subseteq U$;
2. *If* $U_1 \subseteq U_2$ *and* $U_2 \subseteq U_1$, *then* $U_1 = U_2$;
3. *If* $U_1 \subseteq U_2$ *and* $U_2 \subseteq U_3$, *then* $U_1 \subseteq U_3$;
4. $U_1 \cap U_2 \subseteq U_1$ *and* $U_1 \cap U_2 \subseteq U_2$; *and*
5. *If I is a piece of information,* $I \in U_1$ *and* $U_1 \subseteq U_2$, *then* $I \in U_2$.

In establishing the convergence theory for information nets based on closed intervals in [1], the property that the intersection of two closed intervals is still a closed interval played an important role in proofs of various theorems. On the other hand, the fact that the intersection of two open intervals generally is not an open interval has profound consequences: Because of this fact, for convergence of information nets based on open intervals, we cannot generally establish the Moore-Smith style convergence theory in [1] that was based on closed intervals. For convergence of information nets based on open intervals, we can only establish the Moore-Smith style convergence theory in informalogical spaces that satisfy the condition that the intersection of two open intervals is still an open interval. Due to the importance of this condition, we give a special name to informalogical spaces that satisfy the condition.

Definition 2.2. *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space. We say that* $(\mathcal{S}, \mathcal{I})$ *is an open (proper open) normal informalogical space if the intersection of any two open (proper open) intervals is still an open (proper open) interval.*

The expression "... open (proper open) ..." in the above definition means that we can replace "open" by "proper open" to get a parallel definition for the proper open case. We use this type of expressions in various definitions and theorems hereafter. All of the expressions carry similar meanings and we do not explain in each occasion. As mentioned earlier, proper open case is the natural and important thing in the open case. Thus, when we introduce concepts and establish theorems for the open case, we also introduce parallel and independent concepts and establish parallel and independent theorems for the proper open case. The use of the expression "... open (proper open) ..." in definitions and theorems can establish parallel and independent definitions and theorems for both the open and the proper open case, and at the same time, the use of the expression makes the definitions and theorems concise. As mentioned above, we do not explain the use of this type of expressions in each occasion.

By repeatedly applying the property that the intersection of any two open intervals is still an open interval, it is easy to see that the intersection of any finite number of open intervals is still an open interval in an open normal informalogical space. The situation is the same for a proper open normal informalogical space.

Below we introduce a special type of informalogies, *linear* informalogies, and prove that informalogical spaces with linear informalogies are open normal informalogical spaces. Before doing that, we introduce a concept that is used in the next definition: we say that two pieces of information I and J are *comparable* if either $I \preceq J$ or $J \preceq I$.

Definition 2.3. *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space. We say that the informalogy $\mathcal{I}$ is a linear informalogy if any two pieces of information in $\mathcal{I}$ are comparable.*

Theorem 2.1. *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space. If $\mathcal{I}$ is a linear informalogy, then $(\mathcal{S}, \mathcal{I})$ is an open (proper open) normal informalogical space.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

First, we show that the intersection of two proper open intervals is still a proper open interval. Suppose (X_1, Y_1) and (X_2, Y_2) are two proper open intervals in $(\mathcal{S}, \mathcal{I})$. When $(X_1, Y_1) \cap (X_2, Y_2)$ is empty, it is obvious it can be a proper open interval since we can have $(X_1, Y_1) \cap (X_2, Y_2) = (X_1, X_1)$. When $(X_1, Y_1) \cap (X_2, Y_2)$ is non-empty, we show that $(X_1, Y_1) \cap (X_2, Y_2) = (X_1 \vee X_2, Y_1 \wedge Y_2)$.

Since $\mathcal{I}$ is a linear informalogy, either $X_1 \preceq X_2$ or $X_2 \preceq X_1$. Without loss of generality, assume $X_1 \preceq X_2$. Similarly, assume $Y_1 \preceq Y_2$. Then, $X_1 \vee X_2 = X_2$ and $Y_1 \wedge Y_2 = Y_1$. Therefore, $(X_1 \vee X_2, Y_1 \wedge Y_2) = (X_2, Y_1)$.

When $I \in (X_1, Y_1) \cap (X_2, Y_2)$, we have $I \in (X_1, Y_1)$ and $I \in (X_2, Y_2)$. Thus, $X_1 \prec I \prec Y_1$ and $X_2 \prec I \prec Y_2$. Consequently, $X_2 \prec I \prec Y_1$ which means $I \in (X_2, Y_1)$. Thus, $(X_1, Y_1) \cap (X_2, Y_2) \subseteq (X_2, Y_1)$.

On the other hand, when $I \in (X_2, Y_1)$, we have $X_2 \prec I \prec Y_1$. Since $X_1 \preceq X_2$, $X_1 \prec I$, and since $Y_1 \preceq Y_2$, $I \prec Y_2$. Thus, we have $X_1 \prec I \prec Y_1$ and $X_2 \prec I \prec Y_2$, which means $I \in (X_1, Y_1)$ and $I \in (X_2, Y_2)$. Consequently, $I \in (X_1, Y_1) \cap (X_2, Y_2)$. Thus, $(X_2, Y_1) \subseteq (X_1, Y_1) \cap (X_2, Y_2)$.

Now, we know $(X_1, Y_1) \cap (X_2, Y_2) = (X_2, Y_1)$ which means that $(X_1, Y_1) \cap (X_2, Y_2)$ is a proper open interval.

Next, we show that the intersection of two lower special open intervals or two upper special open intervals is still a special open interval. Suppose $[0, Y_1)$ and $[0, Y_2)$ are two lower special open intervals in $(\mathcal{S}, \mathcal{I})$. When $[0, Y_1) \cap [0, Y_2)$ is empty, it is obvious it can be a lower special open interval since we can have $[0, Y_1) \cap [0, Y_2) = [0, 0)$. When $[0, Y_1) \cap [0, Y_2)$ is non-empty, we show that $[0, Y_1) \cap [0, Y_2) = [0, Y_1 \wedge Y_2)$. Since $\mathcal{I}$ is a linear informalogy, without loss of generality, assume $Y_1 \preceq Y_2$. Thus, $Y_1 \wedge Y_2 = Y_1$. Therefore, $[0, Y_1 \wedge Y_2) = [0, Y_1)$.

It is obvious $[0, Y_1) \cap [0, Y_2) \subseteq [0, Y_1)$. On the other hand, when $I \in [0, Y_1)$, we have $I \prec Y_1$. Since $Y_1 \preceq Y_2$, $I \prec Y_2$. Thus, $I \in [0, Y_2)$. (Note that $0 \preceq I$ is true for any piece of information I.) Consequently, $[0, Y_1) \subseteq [0, Y_1) \cap [0, Y_2)$.

Now, we know $[0, Y_1) \cap [0, Y_2) = [0, Y_1)$ which means that $[0, Y_1) \cap [0, Y_2)$ is still a lower special open interval. Similarly, the intersection of two upper special open intervals is still an upper special open interval.

Now consider the intersection of a lower special open interval $[0, Y)$ and an upper special open interval $(X, \Omega]$. We show $[0, Y) \cap (X, \Omega] = (0, Y) \cap (X, \Omega)$. It is obvious that $(0, Y) \cap (X, \Omega) \subseteq [0, Y) \cap (X, \Omega]$. On the other hand, let $I \in [0, Y) \cap (X, \Omega]$. Since $0 \preceq X \prec I$, $0 \prec I$. Since $I \prec Y \preceq \Omega$, $I \prec \Omega$. (Note, that $I \preceq \Omega$ is true for any piece of information I in the informalogical space $(\mathcal{S}, \mathcal{I})$.) These mean that $I \in (0, Y)$ and $I \in (X, \Omega)$. Consequently, $[0, Y) \cap (X, \Omega] \subseteq (0, Y) \cap (X, \Omega)$. Thus, we have $[0, Y) \cap (X, \Omega] = (0, Y) \cap (X, \Omega)$ is a proper open interval.

Finally, we show that the intersection of a special open interval and a proper open interval is still a proper open interval. Let $[0, Y_1)$ be a lower special open interval, and let (X_2, Y_2) be a proper open interval. We show $[0, Y_1) \cap (X_2, Y_2) = (0, Y_1) \cap (X_2, Y_2)$. It is obvious that $(0, Y_1) \cap (X_2, Y_2) \subseteq [0, Y_1) \cap (X_2, Y_2)$. On the other hand, let $I \in [0, Y_1) \cap (X_2, Y_2)$. Since $0 \preceq X_2 \prec I$, $0 \prec I$. Thus, $I \in (0, Y_1)$, and consequently, $[0, Y_1) \cap (X_2, Y_2) \subseteq (0, Y_1) \cap (X_2, Y_2)$. Thus, $[0, Y_1) \cap (X_2, Y_2) = (0, Y_1) \cap (X_2, Y_2)$ is a proper open interval. Similarly, the intersection of an upper special open interval and a proper open interval is still a proper open interval.

This theorem is proved by combining all the facts above. □

After introducing the concept of open intervals, we can introduce the concept of an open neighborhood.

Definition 2.4. *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space, and let I be a piece of information (proper information) in the informalogical space. Let U be a non-empty open (proper open) interval in the informalogical space. When $I \in U$, we say that U is an $\mathcal{I}$-open ($\mathcal{I}$-proper open) neighborhood, or open (proper open) neighborhood for short, of I. We can use $U_{(I)}$ or simply U to denote an open (proper open) neighborhood if no confusion seems possible.*

It is obvious that an open neighborhood of the two special pieces of information 0 and Ω must be a special open interval. More accurately, an open neighborhood of 0 must be a lower special open interval and an open neighborhood of Ω must be an upper special open interval. In fact, the very reason that we include special open intervals in the class of open intervals is that each piece of information in an informalogical space, including 0 and Ω, can have an open neighborhood. The proper open interval $(0, \Omega)$ is a proper open neighborhood of any piece of proper information in an informalogical space.

After open neighborhood is introduced, the neighborhood introduced in [1] (see Definition 1.3 above) can be called *closed neighborhood*. In both Definition 1.3 and Definition 2.4 we say we can use $U_{(I)}$ or U to represent a closed neighborhood and an open neighborhood. In actual usages, usually context is clear enough to tell whether the symbol represents a closed neighborhood or an open neighborhood. When confusions seem possible, we can clarify or simply use complete symbols, such as $U_{(I)}[X, Y]$ for a closed neighborhood and $U_{(I)}(X, Y)$ for an open neighborhood. The situation is similar in other occasions where we use same notations to represent both the closed version and the open version of a concept, and we do not explain this in each occasion.

Definition 2.5. *We say that the family of all open (proper open) neighborhoods of a piece of information (proper information) I is the open (proper open) neighborhood system of I. When no confusion seems possible, we often use $\mathcal{U}_I$ to denote the open (proper open) neighborhood system of I.*

If $\mathcal{U}_0 \subseteq \mathcal{U}_I$, and every open (proper open) neighborhood of I contains a member of $\mathcal{U}_0$ as an open (proper open) subinterval, we say that $\mathcal{U}_0$ is an open (proper open) base for the open (proper open) neighborhood system of I, or an open (proper open) local base at I.

With the *open neighborhood system* and *open local base* introduced, the corresponding concepts of *neighborhood system* and *local base* we introduced in [1] can be called *closed neighborhood system* and *closed local base*.

Since the intersection of any finite number of open intervals is still an open interval in an open normal informalogical space, the following theorem is obvious.

Theorem 2.2. *In an open (proper open) normal informalogical space, the intersection of any finite number of open (proper open) neighborhoods of a piece of information (proper information) is still an open (proper open) neighborhood of that piece of information.*

Next, we prove another theorem about open neighborhood system in an open normal informalogical space.

Theorem 2.3. *Suppose $(\mathcal{S}, \mathcal{I})$ is an open (proper open) normal informalogical space. Let I be a piece of information (proper information) in the space, and let $\mathcal{U}_I$ be its open (proper open) neighborhood system. Then, $(\mathcal{U}_I, \subseteq)$ is a directed set, where $\subseteq$ is the usual subset relation.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Suppose $(\mathcal{S}, \mathcal{I})$ is an open normal informalogical space, and I is a piece of information in the space. When I is a piece of proper information (*i.e.*, $I \neq 0$ and $I \neq \Omega$, $(0, \Omega)$ is an open neighborhood of I. When either $I = 0$ or $I = \Omega$, either $[0, \Omega)$ or $(0, \Omega]$ is a neighborhood of I, respectively. Thus, $\mathcal{U}_I$ is non-empty.

Suppose $U_1, U_2, U_3 \in \mathcal{U}_I$ are such that $U_1 \subseteq U_2$ and $U_2 \subseteq U_3$. Then, by Lemma 2.1, $U_1 \subseteq U_3$. For $U \in \mathcal{U}_I$, also by Lemma 2.1, $U \subseteq U$. Suppose that U_1 and U_2 are two members of $\mathcal{U}_I$, which means that U_1 and U_2 are two open neighborhoods of I. Let $V = U_1 \cap U_2$. Since $(\mathcal{S}, \mathcal{I})$ is an open normal informalogical space, by Theorem 2.2, V is a member of $\mathcal{U}_I$, and by Lemma 2.1, $V \subseteq U_1$ and $V \subseteq U_2$. This means $(\mathcal{U}_I, \subseteq)$ is a directed set □

For later discussions of open convergence of information nets, we also need the concept of product directed set of two directed sets. The following notions and Lemma 2.2 concerning product directed sets all come from [3, Chapter 2].

Suppose $(D_1, \geq_1)$ and $(D_2, \geq_2)$ are two directed sets, and let $D = D_1 \times D_2$ be the Cartesian product of D_1 and D_2. We can construct a binary relation $\geq$ on D as follows: For $(m_1, n_1) \in D$ and $(m_2, n_2) \in D$, $(m_1, n_1) \geq (m_2, n_2)$ if and only if $m_1 \geq_1 m_2$ and $n_1 \geq_2 n_2$. Here, $m_1, m_2 \in D_1$ and $n_1, n_2 \in D_2$. The relation $\geq$ is called the *product order* of $\geq_1$ and $\geq_2$, and in this paper, the product order $\geq$ is also called the *product relation* to indicate the fact that it is a "product" of two binary relations.

Lemma 2.2. ([3, Chapter 2]) *The product relation $\geq$ directs the Cartesian product $D = D_1 \times D_2$, so $(D_1 \times D_2, \geq)$ is a directed set.*

$(D_1 \times D_2, \geq)$ is called the *product directed set* of $(D_1, \geq_1)$ and $(D_2, \geq_2)$.

Using open neighborhoods, we can introduce the concept of open accumulation information.

Definition 2.6. *Let $\mathcal{A}$ be an information set in an informalogical space $(\mathcal{S}, \mathcal{I})$ (i.e., $\mathcal{A} \subseteq \mathcal{S}$). Let I be a piece of information (proper information) in the informalogical space (i.e., $I \in \mathcal{S}$). We say that I is a piece of $\mathcal{I}$-open ($\mathcal{I}$-proper open) accumulation information, or a piece of open (proper open) accumulation information for short, of the information set $\mathcal{A}$ if every open (proper open) neighborhood of I contains a member of $\mathcal{A}$ that is different from I itself.*

With the *open accumulation information* introduced, the corresponding *accumulation information* we introduced in [1] can also be called *closed accumulation information*.

3. Open Convergence of Information Nets

In [1], we introduced the concept of convergence of information nets based on closed neighborhoods. That convergence can be called *closed convergence*. We built a convergence theory that is similar to the Moore-Smith convergence theory in general topology in that, for all major results of Moore-Smith convergence theory (see [3, Chapter 2]), we obtained similar results for the closed convergence of information nets in an informalogical space. However, as shown in Theorem 1.3 of Section 1, the closed convergence has an undesirable property: in an informalogical space $(\mathcal{S}, \mathcal{I})$, if I is in the informalogy (*i.e.*, $I \in \mathcal{I}$), then the necessary condition for an information net to converge to I is that the information net needs to be eventually identical to I.

In this section, we introduce the concept of *open convergence* of information nets based on open neighborhoods. Open convergence avoids the above undesirable property of closed convergence. However, for open convergence, we cannot establish the Moore-Smith style convergence theory for general informalogical spaces as we did for closed convergence. For open convergence, we can only establish the Moore-Smith style convergence theory in open normal informalogical spaces.

Definition 3.1. *Let $\{T_n, n \in D, \geq\}$ be an information net, and let $\mathcal{A}$ be an information set in an informalogical space $(\mathcal{S}, \mathcal{I})$. Then*

1. *we say that the information net $\{T_n, n \in D, \geq\}$ is in the information set $\mathcal{A}$ if $T_n \in \mathcal{A}$ for every $n \in D$;*
2. *we say that the information net $\{T_n, n \in D, \geq\}$ is eventually in the information set $\mathcal{A}$ if there is an $m \in D$ such that $T_n \in \mathcal{A}$ for every $n \in D$ that satisfies $n \geq m$; and*

3. *we say that the information net* $\{T_n, n \in D, \geq\}$ *is frequently in the information set* $\mathcal{A}$ *if, for every* $m \in D$*, there is* $n \in D$ *such that* $n \geq m$ *and* $T_n \in \mathcal{A}$.

Definition 3.2. *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space,* $\{T_n, n \in D, \geq\}$ *be an information net in the space, and* I *be a piece of information (proper information) in the space. We say that the information net* $\{T_n, n \in D, \geq\}$ *open (proper open) converges to* I *in the informalogical space* $(\mathcal{S}, \mathcal{I})$*, or say that* $\{T_n, n \in D, \geq\}$ $\mathcal{I}$*-open (*$\mathcal{I}$*-proper open) converges to* I*, if the information net* $\{T_n, n \in D, \geq\}$ *is eventually in every open (proper open) neighborhood of* I*. The information* I *is called a piece of* $\mathcal{I}$*-open (*$\mathcal{I}$*-proper open) limit information of the information net* $\{T_n, n \in D, \geq\}$ *if* $\{T_n, n \in D, \geq\}$ $\mathcal{I}$*-open (*$\mathcal{I}$*-proper open) converges to* I*. When no confusion would arise, for short, we simply say that the information net* $\{T_n, n \in D, \geq\}$ *open (proper open) converges to information* I*, and that the information* I *is a piece of open (proper open) limit information of the information net* $\{T_n, n \in D, \geq\}$.

It is worthy to point out that an information net may open converge to more than one piece of limit information: Let $\mathcal{S}$ be a non-empty information set that contains 0 and $\Omega = \vee\mathcal{S}$. Let $\mathcal{I} = \{0, \Omega\}$ be the trivial informalogy (see [1]) for the space $\mathcal{S}$. Then, in this informalogical space $(\mathcal{S}, \mathcal{I})$, there are only three open intervals: $(0, \Omega)$, $[0, \Omega)$, and $(0, \Omega]$, where $(0, \Omega)$ is a proper open interval, $[0, \Omega)$ is a lower special open interval and $(0, \Omega]$ is an upper special open interval. Let $\{T_n, n \in D, \geq\}$ be a proper information net in $(\mathcal{S}, \mathcal{I})$ which means that each T_n ($n \in D$) is a piece of proper information in $\mathcal{S}$. Then, it is easy to know that the information net open converges to any piece of information in the space $\mathcal{S}$. This simple example shows that, generally, there is no uniqueness of open (proper open) limit information. However, in an open separated informalogical space to be introduced below, and only in an open separated informalogical space, the open limit information of an information net, if such a limit exists, is unique.

Definition 3.3. *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space. We say that* $(\mathcal{S}, \mathcal{I})$ *is an open (proper open) separated informalogical space, or say that it is open (proper open) separated, if for every two distinct pieces of information (proper information)* I *and* J *in the space, i.e.,* $I, J \in \mathcal{S}$ *and* $I \neq J$*, there exist open (proper open) neighborhoods* U *and* V *of* I *and* J*, respectively, such that* $U \cap V = \phi$.

Once open separated informalogical space is introduced, the "separated informalogical space" introduced in [1] can be called *closed separated informalogical space.*

Lemma 3.1. *Let U and V be two open (proper open) information intervals. Then, $U \cap V \neq \phi$ if and only if there is a piece of information (proper information) $I \in \mathcal{S}$ such that $I \in U$ and $I \in V$.*

This lemma is obvious.

Theorem 3.1. *Suppose informalogical space $(\mathcal{S}, \mathcal{I})$ is an open (proper open) normal informalogical space. Then, $(\mathcal{S}, \mathcal{I})$ is open (proper open) separated if and only if every information net in the space has at most one piece of open (proper open) limit information.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Assume $(\mathcal{S}, \mathcal{I})$ is open separated. Suppose $I, J \in \mathcal{S}$ and $I \neq J$. Then, there are open neighborhoods U and V of I and J, respectively, such that $U \cap V = \phi$. By Lemma 3.1, an information net cannot be eventually in both U and V. Thus, an information net cannot open converge to both I and J.

To establish the converse, assume $(\mathcal{S}, \mathcal{I})$ is not open separated. We construct an information net that converges to two distinct pieces of information. Since $(\mathcal{S}, \mathcal{I})$ is not open separated, there exist two distinct pieces of information I and J in the space such that every open neighborhood U of I intersects every open neighborhood V of J. By Lemma 3.1, we can select a piece of information $T_{(U,V)} \in \mathcal{S}$ such that $T_{(U,V)} \in U$ and $T_{(U,V)} \in V$. Let $\mathcal{U}_I$ be the open neighborhood system of I, and $\mathcal{U}_J$ be the open neighborhood system of J. Since $(\mathcal{S}, \mathcal{I})$ is open normal, by Theorem 2.3, $(\mathcal{U}_I, \subseteq)$ and $(\mathcal{U}_J, \subseteq)$ are two directed sets. By Lemma 2.2, we can denote $(\mathcal{U}_I \times \mathcal{U}_J, \geq)$ as their product directed set. Then, $\{T_{(U,V)}, (U,V) \in \mathcal{U}_I \times \mathcal{U}_J, \geq\}$ is an information net in the space. Next, we show that the information net $\{T_{(U,V)}, (U,V) \in \mathcal{U}_I \times \mathcal{U}_J, \geq\}$ open converges to both I and J.

In fact, for every open neighborhood U of I and every open neighborhood V of J, we have $(U,V) \in \mathcal{U}_I \times \mathcal{U}_J$. For every $(U', V') \in \mathcal{U}_I \times \mathcal{U}_J$ that satisfies $(U', V') \geq (U, V)$, we have $U' \subseteq U$ and $V' \subseteq V$. We know $T_{(U',V')} \in U'$ and $T_{(U',V')} \in V'$ from the selection of $T_{(U',V')}$. Thus, $T_{(U',V')} \in U$ and $T_{(U',V')} \in V$. This means that the information net $\{T_{(U,V)}, (U,V) \in \mathcal{U}_I \times \mathcal{U}_J, \geq\}$ is eventually in every open neighborhood U of I and every open neighborhood V of J. Consequently, the information net open converges to both I and J. □

Next theorem establishes the relationship between a piece of open accumulation information of an information set and open convergence of an information net in an open normal informalogical space.

Theorem 3.2. *Suppose informalogical space* $(\mathcal{S}, \mathcal{I})$ *is an open (proper open) normal informalogical space. Then, a piece of information (proper information)* I *is a piece of open (proper open) accumulation information of an information set* $\mathcal{A}$ *if and only if there exists an information net in* $\mathcal{A} \setminus \{I\}$ *that open (proper open) converges to* I*.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Assume I is a piece of open accumulation information of $\mathcal{A}$. Let $\mathcal{U}_I$ be the open neighborhood system of I. Then, for every $U \in \mathcal{U}_I$, U contains a member of $\mathcal{A}$ other than I itself. We denote this member of $\mathcal{A}$ as T_U. Then, it is clear that $T_U \in \mathcal{A} \setminus \{I\}$ and $T_U \in U$. Since $(\mathcal{S}, \mathcal{I})$ is open normal, by Theorem 2.3, $(\mathcal{U}_I, \subseteq)$ is a directed set. Thus, $\{T_U, U \in \mathcal{U}_I, \subseteq\}$ is an information net in $\mathcal{A} \setminus \{I\}$.

For every open neighborhood U of I, namely $U \in \mathcal{U}_I$, when $V \in \mathcal{U}_I$ that satisfies $V \subseteq U$, we have $T_V \in U$ since $T_V \in V$ and $V \subseteq U$. This means that the information net $\{T_U, U \in \mathcal{U}_I, \subseteq\}$ is eventually in every open neighborhood U of I, and consequently, this information net open converges to I.

To establish the converse, assume that there exists an information net $\{T_n, n \in D, \geq\}$ in $\mathcal{A} \setminus \{I\}$ that open converges to I. Then, for every open neighborhood U of I, there is $p \in D$ such that $T_n \in U$ for every $n \in D$ that satisfies $n \geq p$. Since, by the definition of a directed set, $p \geq p$, we have $T_p \in U$. We know that $T_p \in \mathcal{A} \setminus \{I\}$. Thus, $T_p \neq I$. This means that every open neighborhood of I contains a member of $\mathcal{A}$ other than I itself, and consequently, I is a piece of open accumulation information of $\mathcal{A}$. □

4. Open Convergence of Information Subnets and Information Sequences

We introduced the concepts of *information subnet* and *information sequence* in [1] and discussed closed convergence of information subnets and information sequences. In this section we first review the concepts of information subnet and information sequence, and then prove that the theorems in [1] related to information subnets and information sequences hold for open convergence in open normal informalogical spaces.

Definition 4.1. ([1]) *Let* $\{T_n, n \in D, \geq\}$ *and* $\{R_m, m \in E, \geq_1\}$ *be two information nets. We say that* $\{T_n, n \in D, \geq\}$ *is an information subnet, or subnet for short, of* $\{R_m, m \in E, \geq_1\}$ *if there exists a function* N *on* D *with values in* E *such that*

1. $T = R \bullet N$*, or equivalently,* $T_n = R_{N_n}$ *for each* $n \in D$*, where "*$\bullet$*" is function composition; and*
2. *for each* $m \in E$*, there is a* $p \in D$ *such that if* $n \geq p$*, then,* $N_n \geq_1 m$*.*

Definition 4.2. ([1]) *We say that an information net* $\{T_n, n \in D, \geq\}$ *is an information sequence if there is a bijective map between* D *and the set of positive integers* $\{1, 2, 3, ...\}$ *that preserves the order. That is, suppose* f *is the bijective map from* D *to* $\{1, 2, 3, ...\}$*, then, by "preserves the order" we mean that, for any* $n_1, n_2 \in D$*,* $n_1 \geq n_2$ *in* D *if and only if* $f(n_1) \geq f(n_2)$ *in* $\{1, 2, 3, ...\}$*.*

Definition 4.3. ([1]) *Let* $\{T_n\}$ *and* $\{R_m\}$ *be two information sequences. We say that* $\{T_n\}$ *is an information subsequence, or subsequence for short, of* $\{R_m\}$*, if, viewed as two information nets,* $\{T_n\}$ *is a subnet of* $\{R_m\}$*.*

Without loss of generality, we can write an information sequence as $\{T_1, T_2, ..., T_n, ...\}$. For simplicity, we often simply write $\{T_1, T_2, ..., T_n, ...\}$ as $\{T_n\}$. If $\{T_n\}$ is an information subsequence of $\{R_m\}$, then there is a function N on the positive integers and values in the positive integers such that $T_i = R_{N_i}$ for $i = 1, 2, ..., n, ...$, and for each positive integer m, there is a positive integer n such that if $i \geq n$, then $N_i \geq m$.

In [1], we introduced the concepts of *cluster information* of an information net and *first countable* informalogical space based on closed neighborhoods. Now, we introduce open version of those two concepts based on open neighborhoods.

Definition 4.4. *We say that a piece of information (proper information)* I *is a piece of open (proper open) cluster information of an information net* $\{T_n, n \in D, \geq\}$ *if the information net* $\{T_n, n \in D, \geq\}$ *is frequently in every open (proper open) neighborhood of* I*.*

Definition 4.5. *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space. We say that the informalogical space is open (proper open) first countable if the open (proper open) neighborhood family of each piece of information (proper information) in the space has a countable open (proper open) base. In other words, there is a countable open (proper open) local base at each piece of information (proper information) in the space.*

With the preparation completed we prove that the theorems in [1] related to information subnets and information sequences hold for open convergence in open normal informalogical spaces.

Lemma 4.1. *Let $(\mathcal{S}, \mathcal{I})$ be an open (proper open) normal informalogical space. Suppose that $\{R_m, m \in E, \geq_1\}$ is an information net in the space, I is a piece of information (proper information) in the space, and $\mathcal{U}_I$ is the open (proper open) neighborhood system of I. If the information net is frequently in every member of $\mathcal{U}_I$, then, there is an information subnet of $\{R_m, m \in E, \geq_1\}$ that open (proper open) converges to I.*

PROOF. We only show proof of the open case. Proof of the proper open case is similar.

Let $U \in \mathcal{U}_I$. Since $\{R_m, m \in E, \geq_1\}$ is frequently in U, for each $m_0 \in E$ there is $m \in E$ such that $m \geq_1 m_0$ and $R_m \in U$. Let $D = \{(m, U) | m \in E, U \in \mathcal{U}_I \text{ and } R_m \in U\}$. We define a binary relation $\geq$ on D as follows: $(m_2, U_2) \geq (m_1, U_1)$ if and only if $m_2 \geq_1 m_1$ and $U_2 \subseteq U_1$.

First, we show that $(D, \geq)$ is a directed set. It is obvious that D is non-empty. If $(m_3, U_3) \geq (m_2, U_2)$ and $(m_2, U_2) \geq (m_1, U_1)$ then, we have $m_3 \geq_1 m_2$ and $m_2 \geq_1 m_1$, and $U_3 \subseteq U_2$ and $U_2 \subseteq U_1$. These imply $m_3 \geq_1 m_1$ and $U_3 \subseteq U_1$. Thus, $(m_3, U_3) \geq (m_1, U_1)$. It is clear that $(m, U) \geq (m, U)$, since $m \geq_1 m$ and $U \subseteq U$. For any two members (m_1, U_1) and (m_2, U_2) of D, there is an $n' \in E$ such that $n' \geq_1 m_1$ and $n' \geq_1 m_2$ since $(E, \geq_1)$ is a directed set. Let $V = U_1 \cap U_2$. Then $V \subseteq U_1$ and $V \subseteq U_2$. Since $(\mathcal{S}, \mathcal{I})$ is an open normal informalogical space, by Theorem 2.2, $V \in \mathcal{U}_I$. For $V \in \mathcal{U}_I$ and $n' \in E$, since $\{R_m, m \in E, \geq_1\}$ is frequently in V, there is an $n \in E$ such that $n \geq_1 n'$ and $R_n \in V$. Then, we know 1) $(n, V) \in D$; 2) $(n, V) \geq (m_1, U_1)$ since $n \geq_1 n' \geq_1 m_1$ and $V \subseteq U_1$; and 3) $(n, V) \geq (m_2, U_2)$ since $n \geq_1 n' \geq_1 m_2$ and $V \subseteq U_2$. Now we know that $(D, \geq)$ is a directed set.

Second, we construct a subnet of $\{R_m, m \in E, \geq_1\}$ as follows: For each $(m, U) \in D$, let $N_{(m,U)} = m$ and $T_{(m,U)} = R_m$. Then, $T = R \bullet N$, where

"$\bullet$" is function composition. We construct an open neighborhood U_0 of I as follows: if I is a piece of proper information, then $U_0 \equiv (0, \Omega)$; and if $I = 0$ or Ω, then $U_0 \equiv [0, \Omega)$ or $U_0 \equiv (0, \Omega]$, respectively. Then, for each $n \in E$ and the open neighborhood U_0 of I, there is $p \in E$ such that $p \geq_1 n$ and $R_p \in U_0$, since the information net $\{R_m, m \in E, \geq_1\}$ is frequently in U_0. Then, $(p, U_0) \in D$. For any $(m, U) \in D$ that satisfies $(m, U) \geq (p, U_0)$, we have $N_{(m,U)} = m \geq_1 p \geq_1 n$. This shows that $\{T_{(m,U)}, (m, U) \in D, \geq\}$ is a subnet of $\{R_m, m \in E, \geq_1\}$.

Finally, we show that the information net $\{T_{(m,U)}, (m, U) \in D, \geq\}$ open converges to I. For every open neighborhood $V \in \mathcal{U}_I$, since $\{R_m, m \in E, \geq_1\}$ is frequently in V, there is a $p \in E$ such that $R_p \in V$. Thus, $(p, V) \in D$. If $(m, U) \in D$ and $(m, U) \geq (p, V)$, then $T_{(m,U)} = R_m \in U \subseteq V$, and consequently, $T_{(m,U)} \in V$. This means that the information net $\{T_{(m,U)}, (m, U) \in D, \geq\}$ is eventually in every open neighborhood V of I. Thus, the information net $\{T_{(m,U)}, (m, U) \in D, \geq\}$ open converges to I. □

Theorem 4.1. *Let $(\mathcal{S}, \mathcal{I})$ be an open (proper open) normal informalogical space. Then, a piece of information (proper information) I is a piece of open (proper open) cluster information of an information net $\{R_m, m \in E, \geq_1\}$ if and only if the information net has a subnet that open (proper open) converges to I.*

PROOF. We only show proof of the open case. Proof of the proper open case is similar.

Suppose that I is a piece of open cluster information of $\{R_m, m \in E, \geq_1\}$. Then, $\{R_m, m \in E, \geq_1\}$ is frequently in every open neighborhood of I. Since $(\mathcal{S}, \mathcal{I})$ is open normal, by Lemma 4.1, the information net has a subnet that open converges to I.

To establish the converse, suppose that $\{R_m, m \in E, \geq_1\}$ has a subnet $\{T_n, n \in D, \geq\}$ that open converges to I. Then, for every open neighborhood U of I, there is a $p_1 \in D$ such that if $n \in D$ and $n \geq p_1$, then $T_n \in U$. Since $\{T_n, n \in D, \geq\}$ is a subnet of $\{R_m, m \in E, \geq_1\}$, for every $m \in E$, there is a $p_2 \in D$ such that if $n \in D$ and $n \geq p_2$, then $N_n \geq_1 m$, where N is a function on D with values in E and $T = R \bullet N$, or equivalently, $T_n = R_{N_n}$ for each $n \in D$ (see Definition 4.1).

Let us go back to D. Since $(D, \geq)$ is a directed set, for $p_1 \in D$ and $p_2 \in D$, there is a $q \in D$ such that $q \geq p_1$ and $q \geq p_2$. We have $T_q \in U$ since $q \geq p_1$, and we have $N_q \geq_1 m$ since $q \geq p_2$. Let $m' = N_q$. Then, we have

$m^{'} \in E$, $m^{'} \geq_1 m$ and $R_{m'} = R_{N_q} = T_q \in U$. That means, for every open neighborhood U of I and every $m \in E$, there is an $m^{'} \in E$ such that $m^{'} \geq_1 m$ and $R_{m'} \in U$. Thus, $\{R_m, m \in E, \geq_1\}$ is frequently in U. Consequently, I is a piece of open cluster information of the information net $\{R_m, m \in E, \geq_1\}$. $\square$

Theorem 4.2. *Let $(\mathcal{S}, \mathcal{I})$ be an open (proper open) normal informalogical space. Suppose that the informalogical space $(\mathcal{S}, \mathcal{I})$ is open (proper open) first countable, I is a piece of information (proper information) in the space, $\mathcal{A}$ is an information set in the space, and $\{R_m\}$ is an information sequence in the space. Then*

1. *I is a piece of open (proper open) accumulation information of $\mathcal{A}$ if and only if there is an information sequence in $\mathcal{A} \setminus \{I\}$ that open (proper open) converges to I; and*
2. *I is a piece of open (proper open) cluster information of $\{R_m\}$ if and only if $\{R_m\}$ has a subsequence that open (proper open) converges to I.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Since $(\mathcal{S}, \mathcal{I})$ is open first countable, we can assume that $\{U_1, U_2, ..., U_n, ...\}$ is an open local base at I. Then, for $n = 1, 2, 3, ...$, we define $V_n \equiv \cap_{i=1}^{n} U_i$. Since $(\mathcal{S}, \mathcal{I})$ is open normal, by Theorem 2.2, $V_1, V_2, ..., V_n, ...$ are also open neighborhoods of I. It is obvious that $... \subseteq V_{n+1} \subseteq V_n \subseteq ... \subseteq V_2 \subseteq V_1$. Also, $V_n \subseteq U_n$ for $n = 1, 2, 3, ...$ Thus, $\{V_1, V_2, ..., V_n, ...\}$ is also an open local base at I.

1. Suppose I is a piece of open accumulation information of $\mathcal{A}$. Then, for each V_n, there is a $T_n \in \mathcal{A} \setminus \{I\}$ such that $T_n \in V_n$, and consequently, we obtain an information sequence $\{T_1, T_2, ..., T_n, ...\}$ in $\mathcal{A} \setminus \{I\}$. For every open neighborhood U of I, since $\{V_1, V_2, ..., V_n, ...\}$ is an open local base at I, there is a V_p such that $V_p \subseteq U$. Then, when $n \geq p$, we have $T_n \in V_n \subseteq V_p \subseteq U$, and thus $T_n \in U$. This means that the information sequence $\{T_1, T_2, ..., T_n, ...\}$ is eventually in every open neighborhood of I. Therefore, $\{T_1, T_2, ..., T_n, ...\}$ open converges to I. Since an information sequence is also an information net, the converse part of the assertion is established by Theorem 3.2.

2. Suppose I is a piece of open cluster information of the information sequence $\{R_m\}$. Then, for each V_i, there is a positive integer N_i such that $N_i \geq i$ and $R_{N_i} \in V_i$. Let $T_i = R_{N_i}$ for $i = 1, 2, 3, ...$ It is clear that $\{T_n\}$ is a subsequence of $\{R_m\}$. For every open neighborhood U of I, since $\{V_1, V_2, ..., V_n, ...\}$ is an open local base at I, there is a V_p such that $V_p \subseteq U$. Then, when $n \geq p$, we have $T_n = R_{N_n} \in V_n \subseteq V_p \subseteq U$, so $T_n \in U$. This means that the information sequence $\{T_1, T_2, ..., T_n, ...\}$ is eventually in every open neighborhood of I. Therefore, $\{T_1, T_2, ..., T_n, ...\}$ open converges to I.
   Since an information subsequence is also an information subnet, the converse part of the assertion is established by Theorem 4.1. □

5. Open Compactness

In [2], based on the concept of closed interval, we introduced the concept of *interval cover* of an information set in an informalogical space, and based on the concept of interval cover, we introduced the concept of a *compact* informalogical space. In this section, based on the concept of open interval, we introduce the concept of *open interval cover* of an information set in an informalogical space, and based on the concept of open interval cover, we introduce the concept of an *open compact* informalogical space. Then we show 1) an informalogical space $(\mathcal{S}, \mathcal{I})$ is open compact if and only if each information net in the informalogical space has a piece of open cluster information, and 2) if $(\mathcal{S}, \mathcal{I})$ further is an open normal informalogical space, then $(\mathcal{S}, \mathcal{I})$ is open compact if and only if each information net in the informalogical space has a subnet that open converges.

Definition 5.1. *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space, let $\mathcal{A}$ be a set of information (proper information) in the informalogical space (i.e., $\mathcal{A} \subseteq \mathcal{S}$), and let $\mathcal{U}$ be a family of open (proper open) intervals in the informalogical space. We say that the open (proper open) interval family $\mathcal{U}$ is an open (proper open) interval cover of the information (proper information) set $\mathcal{A}$ if $\mathcal{A} \subseteq \cup\mathcal{U}$, or in other words, each piece of information (proper information) in $\mathcal{A}$ is in some open (proper open) interval in the open (proper open) interval family $\mathcal{U}$.*

Definition 5.2. *We say that an informalogical space $(\mathcal{S}, \mathcal{I})$ is an open (proper open) compact informalogical space, or $(\mathcal{S}, \mathcal{I})$ is open (proper open) compact,* *if each open (proper open) interval cover* $\mathcal{U}$ *of* $\mathcal{S}$ *(*$\mathcal{S} \setminus \{0, \Omega\}$*) has a finite open (proper open) subcover, or in other words, there are finite members* U_1, U_2, *...,* U_n *of* $\mathcal{U}$ *such that* $\mathcal{S} \subseteq \cup_{i=1}^{n} U_i$ *(* $\mathcal{S} \setminus \{0, \Omega\} \subseteq \cup_{i=1}^{n} U_i$ *).*

Theorem 5.1. *An informalogical space* $(\mathcal{S}, \mathcal{I})$ *is open (proper open) compact if and only if each information net in the informalogical space has a piece of open (proper open) cluster information.*

PROOF. We only show proof of the open case. Proof of the proper open case is similar.

Suppose that $(\mathcal{S}, \mathcal{I})$ is an open compact informalogical space, and suppose that $\{T_n, n \in D, \geq\}$ is an information net in the informalogical space. If the information net $\{T_n, n \in D, \geq\}$ has no open cluster information, then, for each piece of information I in the space $\mathcal{S}$ (*i.e.*, $I \in \mathcal{S}$), there is an open neighborhood U_I of I (*i.e.*, U_I is an open information interval and $I \in U_I$) and $n_I \in D$ such that $T_n \notin U_I$ for $n \in D$ and $n \geq n_I$. Let $\mathcal{U} \equiv \{U_I | I \in \mathcal{S}\}$, then $\mathcal{U}$ is an open interval cover of $\mathcal{S}$. Since $(\mathcal{S}, \mathcal{I})$ is open compact, $\mathcal{U}$ has a finite open subcover $\{U_{I1}, U_{I2}, ..., U_{Ir}\}$. Since D is a directed set, there is $n_0 \in D$ such that $n_0 \geq n_{Ij}$, $j = 1, 2, ..., r$. Therefore, $T_{n0} \notin U_{Ij}$, $j = 1, 2, ..., r$. However, this is contradictory to that $\{U_{I1}, U_{I2}, ..., U_{Ir}\}$ is an open interval cover of the space $\mathcal{S}$. This proves that if $(\mathcal{S}, \mathcal{I})$ is open compact then each information net in the informalogical space has a piece of open cluster information.

Conversely, suppose that each information net in $(\mathcal{S}, \mathcal{I})$ has a piece of open cluster information, and suppose that $\mathcal{U}$ is an open interval cover of $\mathcal{S}$. We show that $\mathcal{U}$ has a finite subcover. Otherwise, for each finite subfamily $n \equiv \{U_1, U_2, ..., U_r\}$ of $\mathcal{U}$ (*i.e.*, $U_j \in \mathcal{U}, j = 1, 2, ..., r$), there is $T_n \in \mathcal{S}$ such that $T_n \notin U_j, j = 1, 2, ..., r$. Let $D \equiv \{n | n \text{ is a finite subfamily of } \mathcal{U}\}$. We show that $(D, \supseteq)$ is a directed set, where $\supseteq$ is the regular superset relation between two sets. For $n_1, n_2, n_3 \in D$, if $n_1 \supseteq n_2$ and $n_2 \supseteq n_3$, then $n_1 \supseteq n_3$. A set n is a superset of itself, meaning $n \supseteq n$. For any two members $n_1, n_2 \in D$, let $n = n_1 \cup n_2$. Then, since n_1 and n_2 are two finite subfamilies of $\mathcal{U}$, n is also a finite subfamily of $\mathcal{U}$ and thus $n \in D$. It is obvious that $n \supseteq n_1$ and $n \supseteq n_2$. Now, we know that $(D, \supseteq)$ is a directed set. Therefore, $\{T_n, n \in D, \supseteq\}$ is an information net in the informalogical space $(\mathcal{S}, \mathcal{I})$.

Since each information net in $(\mathcal{S}, \mathcal{I})$ has a piece of open cluster information, let T be a piece of open cluster information of the information net $\{T_n, n \in D, \supseteq\}$. Since $\mathcal{U}$ is an open interval cover of $\mathcal{S}$, there is $U_T \in \mathcal{U}$

such that $T \in U_T$, which means that U_T is an open neighborhood of T. Let $n_0 \equiv \{U_T\} \in D$. Then, since T is a piece of open cluster information of $\{T_n, n \in D, \supseteq\}$, there is an $n \in D$ such that $n \supseteq n_0$ and T_n is in the open neighborhood U_T of T, or in other words, $T_n \in U_T$. However, on the other hand, $n \supseteq n_0$ means $n \supseteq \{U_T\}$ which implies $U_T \in n$. Thus, by the choice of T_n, T_n is not in any open interval in n, which implies $T_n \notin U_T$. Now, we have $T_n \in U_T$ and $T_n \notin U_T$ at the same time. This contradiction shows that $\mathcal{U}$ must have a finite subcover. □

It should be noted that the above theorem does not require $(\mathcal{S}, \mathcal{I})$ to be an open normal informalogical space. In Section 4, Theorem 4.1 shows that, in an open normal informalogical space, a piece of information is a piece of open cluster information of an information net if and only if the information net has a subnet that open converges to that piece of information. Thus, for an open normal informalogical space, we have the following corollary.

Corollary 5.1. *Let $(\mathcal{S}, \mathcal{I})$ be an open (proper open) normal informalogical space. Then, $(\mathcal{S}, \mathcal{I})$ is open (proper open) compact if and only if each information net in the informalogical space has a subnet that open (proper open) converges to a piece of information (proper information) in $(\mathcal{S}, \mathcal{I})$.*

6. Isomorphic Invariants

In [2], we introduced the concept of *isomorphism* between two informalogical spaces. Basically, an isomorphism between two informalogical spaces $(\mathcal{S}_1, \mathcal{I}_1)$ and $(\mathcal{S}_2, \mathcal{I}_2)$ is an order-preserving bijective function f from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, and f preserves the informalogy. That is, 1) f is a bijective function from $\mathcal{S}_1$ to $\mathcal{S}_2$; 2) $f(I) \preceq f(J)$ in $\mathcal{S}_2$ if and only if $I \preceq J$ in $\mathcal{S}_1$ (order-preserving); and 3) $f(I) \in \mathcal{I}_2$ if and only if $I \in \mathcal{I}_1$. An obvious fact about isomorphism is that if f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, then f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$. This fact will be used frequently when we prove various theorems below.

In [2], we also introduced the concept of *isomorphic invariant* which is a property of an informalogical space that is preserved under isomorphisms. In this section, we prove that open limit uniqueness, open normality, open separatedness, open first countability and open compactness are all isomorphic invariants.

The following lemma is a property of order-preserving bijective functions.

Lemma 6.1. *Let f be an order-preserving bijective function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then, $f(I) \prec f(J)$ in $\mathcal{S}_2$ if and only if $I \prec J$ in $\mathcal{S}_1$.*

This lemma is easy to prove: Suppose $I \prec J$. Then, $f(I) \preceq f(J)$ since f is order-preserving. Also, $f(I) = f(J)$ cannot be true since it implies $I = J$ due to that f is bijective. Thus, $f(I) \prec f(J)$. Similarly $f(I) \prec f(J)$ implies $I \prec J$.

Theorem 6.1. *Let f be an order-preserving bijective function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Let $A, B \in \mathcal{S}_1$. Then,*

1. $f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \prec B\}) = \{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \prec f(B)\}$;
2. $f(\{I | I \in \mathcal{S}_1 \text{ and } A \preceq I \prec B\}) = \{J | J \in \mathcal{S}_2 \text{ and } f(A) \preceq J \prec f(B)\}$; *and*
3. $f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \preceq B\}) = \{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \preceq f(B)\}$.

PROOF. 1. Let $X \in f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \prec B\})$. Then, there is $I \in \mathcal{S}_1$ satisfying $A \prec I \prec B$ such that $f(I) = X$. Since f is an order-preserving bijective function, by Lemma 6.1, $f(A) \prec f(I) \prec f(B)$. This means $f(A) \prec X \prec f(B)$. Thus, $X \in \{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \prec f(B)\}$. This also shows that if $f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \prec B\})$ is non-empty then $\{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \prec f(B)\}$ is non-empty.

On the other hand, let $X \in \{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \prec f(B)\}$. Then, $f(A) \prec X \prec f(B)$. Since f is a bijective function, there is $I \in \mathcal{S}_1$ such that $f(I) = X$. Thus, $f(A) \prec f(I) \prec f(B)$. Since f is an order-preserving bijective function, again by Lemma 6.1, $A \prec I \prec B$. That is, $X \in f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \prec B\})$. This also shows that if $\{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \prec f(B)\}$ is non-empty then $f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \prec B\})$ is non-empty.

Combination of the above two facts shows 1) if both $f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \prec B\})$ and $\{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \prec f(B)\}$ are non-empty, then $f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \prec B\}) = \{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \prec f(B)\}$; and 2) if one of $f(\{I | I \in \mathcal{S}_1 \text{ and } A \prec I \prec B\})$ and $\{J | J \in \mathcal{S}_2 \text{ and } f(A) \prec J \prec f(B)\}$ is empty so is the other, and thus the equality is still true.

2. and 3. can be proved similarly. □

Theorem 6.2. *Let f be an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, and let $X, Y \in \mathcal{I}_1$. Then, $f(X), f(Y) \in \mathcal{I}_2$ and*

1. $f((X, Y)) = (f(X), f(Y))$;

2. $f([0, Y)) = [0, f(Y))$; *and*
3. $f((X, \Omega_1]) = (f(X), \Omega_2]$, *where* $\Omega_1 = \vee \mathcal{S}_1$ *and* $\Omega_2 = \vee \mathcal{S}_2$.

Proof. $X, Y \in \mathcal{I}_1$ implies $f(X), f(Y) \in \mathcal{I}_2$ since f is an isomorphism. Also, since an isomorphism is an order-preserving bijective function, by Theorem 4.1 of [2], $f(0) = 0$ and $f(\Omega_1) = \Omega_2$. Then, the three equalities in this theorem come directly from the three corresponding equalities in Theorem 6.1. □

The above theorem basically shows open intervals are preserved under an isomorphism. The following theorem shows open neighborhoods of a piece of information are preserved under an isomorphism.

Theorem 6.3. *Let f be an order-preserving bijective function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then, the following four statements are equivalent.*

1. *f is an isomorphism.*
2. *$(f(X), f(Y))$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if (X, Y) is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$; $(f(X), \Omega_2]$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if $(X, \Omega_1]$ is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$; and $[0, f(Y))$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if $[0, Y)$ is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$.*
3. *$(f(X), \Omega_2]$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if $(X, \Omega_1]$ is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$.*
4. *$[0, f(Y))$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if $[0, Y)$ is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$.*

Proof. We know that f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$ if f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Also, since f is an order-preserving bijective function, by Theorem 4.1 of [2], $f(0) = 0$ and $f(\Omega_1) = \Omega_2$.

Suppose 1. is true, we prove 2.

Assume (X, Y) is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$, then, $X, Y \in \mathcal{I}_1$ and $X \prec I \prec Y$. Since f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, $f(X), f(Y) \in \mathcal{I}_2$, and by Lemma 6.1, $f(X) \prec f(I) \prec f(Y)$. This means that $(f(X), f(Y))$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Conversely, assume $(f(X), f(Y))$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, then by what is just proved,

$(f^{-1}(f(X)), f^{-1}(f(Y))) = (X, Y)$ is an open neighborhood of $f^{-1}(f(I)) = I$ in $(\mathcal{S}_1, \mathcal{I}_1)$.

Noting $f(0) = 0$ and $f(\Omega_1) = \Omega_2$, we can similarly prove that $[0, f(Y))$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if $[0, Y)$ is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$, and $(f(X), \Omega_2]$ is an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if $(X, \Omega_1]$ is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$.

Suppose 2. is true, then it is obvious that 3. and 4. are true.

Suppose 3. is true. We prove 1 is true, which means we prove that $f(I) \in \mathcal{I}_2$ if and only if $I \in \mathcal{I}_1$. Assume $I \in \mathcal{I}_1$. If $I = \Omega_1$, then $f(I) = \Omega_2 \in \mathcal{I}_2$. If $I \neq \Omega_1$, then $f(I) \neq \Omega_2$. $(I, \Omega_1]$ is an open neighborhood of Ω_1 in $(\mathcal{S}_1, \mathcal{I}_1)$ since $I \neq \Omega_1$. Then, by 3., $(f(I), \Omega_2]$ is an open neighborhood of $f(\Omega_1) = \Omega_2$ in $(\mathcal{S}_2, \mathcal{I}_2)$, which implies $f(I) \in \mathcal{I}_2$. Conversely, assume $f(I) \in \mathcal{I}_2$. If $f(I) = \Omega_2$, then $I = \Omega_1 \in \mathcal{I}_1$. If $f(I) \neq \Omega_2$, then $I \neq \Omega_1$. $(f(I), \Omega_2]$ is an open neighborhood of Ω_2 in $(\mathcal{S}_2, \mathcal{I}_2)$ since $f(I) \neq \Omega_2$. Then, by 3., $(I, \Omega_1]$ is an open neighborhood of Ω_1 in $(\mathcal{S}_1, \mathcal{I}_1)$, which implies $I \in \mathcal{I}_1$.

Suppose 4. is true. We can similarly prove 1 is true just by replacing Ω_1 and Ω_2 with 0 and replacing the upper special open intervals with lower special open intervals in the above proof.

Finally we know the four statements are equivalent by combining all the facts proved above. $\square$

We can have the following theorem by applying Theorem 6.2 and Theorem 6.3.

Theorem 6.4. *Let f be an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then we have*

1. *if U is an open (proper open) interval in $(\mathcal{S}_1, \mathcal{I}_1)$, then $f(U)$ is an open (proper open) interval in $(\mathcal{S}_2, \mathcal{I}_2)$; and*
2. *if U is an open (proper open) neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$, then $f(U)$ is an open (proper open) neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$.*

Let f be a function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Let $\{T_n, n \in D, \geq\}$ be an information net in $(\mathcal{S}_1, \mathcal{I}_1)$. Then, $\{f(T_n), n \in D, \geq\}$ is an information net in $(\mathcal{S}_2, \mathcal{I}_2)$. The following theorem establishes the relationship between open convergence of the two information nets in the two informalogical spaces under an isomorphism.

Theorem 6.5. *Let f be an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. The information net $\{f(T_n), n \in D, \geq\}$ open (proper open) converges to $f(T)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if the information net $\{T_n, n \in D, \geq\}$ open (proper open) converges to T in $(\mathcal{S}_1, \mathcal{I}_1)$.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Assume that $\{T_n, n \in D, \geq\}$ open converges to T in $(\mathcal{S}_1, \mathcal{I}_1)$. We consider $\{f(T_n), n \in D, \geq\}$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Let V be an open neighborhood of $f(T)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ and let $U = f^{-1}(V)$. By Theorem 6.4, U is an open neighborhood of T in $(\mathcal{S}_1, \mathcal{I}_1)$ since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$. Since $\{T_n, n \in D, \geq\}$ open converges to T in $(\mathcal{S}_1, \mathcal{I}_1)$, there is $p \in D$ such that $T_n \in U$ for $n \in D$ satisfying $n \geq p$. $T_n \in U$ means that U is an open neighborhood of T_n in $(\mathcal{S}_1, \mathcal{I}_1)$. By Theorem 6.4, $f(U) = V$ is an open neighborhood of $f(T_n)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. This means $f(T_n) \in V$ for $n \in D$ satisfying $n \geq p$. Thus, $\{f(T_n), n \in D, \geq\}$ open converges to $f(T)$ in $(\mathcal{S}_2, \mathcal{I}_2)$.

Conversely, assume that the information net $\{f(T_n), n \in D, \geq\}$ open converges to $f(T)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Then, since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, by what is just proved, the information net $\{f^{-1}(f(T_n)), n \in D, \geq \} = \{T_n, n \in D, \geq\}$ open converges to $f^{-1}(f(T)) = T$ in $(\mathcal{S}_1, \mathcal{I}_1)$. □

Theorem 6.6. *Let f be an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, let $\mathcal{A}$ be an information set in $(\mathcal{S}_1, \mathcal{I}_1)$ and let I be a piece of information (proper information) in $(\mathcal{S}_1, \mathcal{I}_1)$. Then, $f(I)$ is an open (proper open) accumulation information of the information set $f(\mathcal{A})$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if I is an open (proper open) accumulation information of the information set $\mathcal{A}$ in $(\mathcal{S}_1, \mathcal{I}_1)$.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Since f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$.

Suppose I is an open accumulation information of the information set $\mathcal{A}$ in $(\mathcal{S}_1, \mathcal{I}_1)$. We prove $f(I)$ is an open accumulation information of the information set $f(\mathcal{A})$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Let V be an open neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, by Theorem 6.4, $f^{-1}(V)$ is an open neighborhood of $f^{-1}(f(I)) = I$ in $(\mathcal{S}_1, \mathcal{I}_1)$. Since I is an open accumulation information of the information set $\mathcal{A}$ in $(\mathcal{S}_1, \mathcal{I}_1)$, there is $A \in \mathcal{A}$ such that $A \neq I$ and $A \in f^{-1}(V)$. Then, since f is a bijective function, $f(A) \in f(\mathcal{A})$, $f(A) \neq f(I)$ and $f(A) \in f(f^{-1}(V)) = V$. This shows that every open neighborhood of $f(I)$ contains a member of $f(\mathcal{A})$ that is different from $f(I)$ itself. Thus, $f(I)$ is an open accumulation information of the information set $f(\mathcal{A})$ in $(\mathcal{S}_2, \mathcal{I}_2)$.

Since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, $f^{-1}(f(I)) = I$ and $f^{-1}(f(\mathcal{A})) = \mathcal{A}$, the converse part of this theorem is obtained by what is just proved. □

Next we present theorems about isomorphic invariants in cases of concepts based on open intervals. An isomorphic invariant is a property of an informalogical space that is preserved under isomorphisms. (See [2]).

Theorem 6.7. *Open (proper open) limit uniqueness is an isomorphic invariant.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Let $(\mathcal{S}_1, \mathcal{I}_1)$ be an informalogical space in which an information net has at most one piece of open limit information. Let $(\mathcal{S}_1, \mathcal{I}_1)$ be isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$. We prove that an information net in $(\mathcal{S}_2, \mathcal{I}_2)$ also has at most one piece of open limit information.

Suppose f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then, f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$. Suppose $\{T_n, n \in D, \geq\}$ is an information net in $(\mathcal{S}_2, \mathcal{I}_2)$. If $\{T_n, n \in D, \geq\}$ has two distinct pieces of open limit information I and J $(I \neq J)$ in $(\mathcal{S}_2, \mathcal{I}_2)$, then, since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, by Theorem 6.5, the information net $\{f^{-1}(T_n), n \in D, \geq\}$ in $(\mathcal{S}_1, \mathcal{I}_1)$ has two distinct pieces of open limit information $f^{-1}(I)$ and $f^{-1}(J)$ $(f^{-1}(I) \neq f^{-1}(J))$ in $(\mathcal{S}_1, \mathcal{I}_1)$. However, this is contradictory to the open limit uniqueness in $(\mathcal{S}_1, \mathcal{I}_1)$. This proves that $\{T_n, n \in D, \geq\}$ can only have at most one piece of open limit information.
□

Theorem 6.8. *Open (proper open) normality is an isomorphic invariant.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Let $(\mathcal{S}_1, \mathcal{I}_1)$ be an open normal informalogical space, and let $(\mathcal{S}_1, \mathcal{I}_1)$ be isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$. We prove that $(\mathcal{S}_2, \mathcal{I}_2)$ is also an open normal informalogical space. In other words, we prove the intersection $U \cap V$ of two open intervals U and V in $(\mathcal{S}_2, \mathcal{I}_2)$ is also an open interval.

Suppose f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then, f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, and consequently f^{-1} is an order-preserving bijective function. If $U \cap V$ is empty then it is a trivial open interval. Suppose $U \cap V$ is non-empty and $I \in U \cap V$. Then, since f^{-1} is a bijective function, $f^{-1}(U \cap V)$ is non-empty, $f^{-1}(I) \in f^{-1}(U \cap V)$, and $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$. Since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$ and U and V are two open intervals in $(\mathcal{S}_2, \mathcal{I}_2)$, by Theorem 6.4, $f^{-1}(U)$ and $f^{-1}(V)$ are two open intervals in $(\mathcal{S}_1, \mathcal{I}_1)$. Then, $f^{-1}(U) \cap f^{-1}(V)$ and consequently $f^{-1}(U \cap V)$ is an open interval in $(\mathcal{S}_1, \mathcal{I}_1)$ since $(\mathcal{S}_1, \mathcal{I}_1)$ is open normal. Again by Theorem 6.4, $f(f^{-1}(U \cap V)) = U \cap V$ is an open interval in $(\mathcal{S}_2, \mathcal{I}_2)$. This proves the normality of $(\mathcal{S}_2, \mathcal{I}_2)$. □

Theorem 6.9. *Open (proper open) separatedness is an isomorphic invariant.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Let $(\mathcal{S}_1, \mathcal{I}_1)$ be an open separated informalogical space, and let $(\mathcal{S}_1, \mathcal{I}_1)$ be isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$. We prove that $(\mathcal{S}_2, \mathcal{I}_2)$ is also open separated.

Let J_1 and J_2 be two distinct pieces of information in $(\mathcal{S}_2, \mathcal{I}_2)$, or in other words, $J_1, J_2 \in \mathcal{S}_2$ and $J_1 \neq J_2$. Suppose f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$ which implies that f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$. Then, $f^{-1}(J_1), f^{-1}(J_2) \in \mathcal{S}_1$ and $f^{-1}(J_1) \neq f^{-1}(J_2)$. Since $(\mathcal{S}_1, \mathcal{I}_1)$ is open separated, there are open neighborhoods U_1 and U_2 of $f^{-1}(J_1)$ and $f^{-1}(J_2)$, respectively, in $(\mathcal{S}_1, \mathcal{I}_1)$ such that $U_1 \cap U_2 = \phi$. By Theorem 6.4, $f(U_1)$ and $f(U_2)$ are neighborhoods of $f(f^{-1}(J_1)) = J_1$ and $f(f^{-1}(J_2)) = J_2$, respectively, in $(\mathcal{S}_2, \mathcal{I}_2)$.

Next, we show that $f(U_1) \cap f(U_2) = \phi$. Otherwise, there is $J \in f(U_1) \cap f(U_2) \subseteq \mathcal{S}_2$. Then, there is $I \in \mathcal{S}_1$ such that $f(I) = J$. $J \in f(U_1)$ and $J \in f(U_2)$ imply that $f(U_1)$ and $f(U_2)$ are open neighborhoods of J. Thus, by Theorem 6.4, $f^{-1}(f(U_1)) = U_1$ and $f^{-1}(f(U_2)) = U_2$ are open neighborhoods of $f^{-1}(J) = f^{-1}(f(I)) = I$. Consequently, $I \in U_1$ and $I \in U_2$. However, this is contradictory to $U_1 \cap U_2 = \phi$. Thus, we must have $f(U_1) \cap f(U_2) = \phi$. This proves that $(\mathcal{S}_2, \mathcal{I}_2)$ is open separated. □

Theorem 6.10. *Open (proper open) first countability is an isomorphic invariant.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Let $(\mathcal{S}_1, \mathcal{I}_1)$ be an open first countable informalogical space, and let $(\mathcal{S}_1, \mathcal{I}_1)$ be isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$. We prove that $(\mathcal{S}_2, \mathcal{I}_2)$ is also open first countable, or in other words, each piece of information J in $\mathcal{S}_2$ has a countable open local base.

Suppose f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$. There is $I \in \mathcal{S}_1$ such that $f(I) = J$. Since $(\mathcal{S}_1, \mathcal{I}_1)$ is open first countable, I has a countable open local base $U_1, U_2, ..., U_n, ...$, where U_i (i=1, 2, ..., n, ...) is an open neighborhood of I. We show that $f(U_1), f(U_2), ..., f(U_n), ...$ is an open local base at J in $(\mathcal{S}_2, \mathcal{I}_2)$.

By Theorem 6.4, $f(U_i)$ (i=1, 2, ..., n, ...) is an open neighborhood of $f(I) = J$. Thus, $f(U_1), f(U_2), ..., f(U_n), ...$ is a subset of the open neighborhood system of J. Let V be an open neighborhood of J. Then, since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, by Theorem 6.4, $f^{-1}(V)$ is an open neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$. Since $U_1, U_2, ..., U_n, ...$ is an open local base at I, there is U_i such that $U_i \subseteq f^{-1}(V)$. Since f is a bijective function, $f(U_i) \subseteq f(f^{-1}(V)) = V$. This proves that $f(U_1), f(U_2), ..., f(U_n), ...$ is an open local base at J. □

Theorem 6.11. *Open (proper open) compactness is an isomorphic invariant.*

Proof. We only show proof of the open case. Proof of the proper open case is similar.

Let $(\mathcal{S}_1, \mathcal{I}_1)$ be an open compact informalogical space, and let $(\mathcal{S}_1, \mathcal{I}_1)$ be isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$. We prove that $(\mathcal{S}_2, \mathcal{I}_2)$ is also open compact, or in other words, each open interval cover $\mathcal{U}$ of $\mathcal{S}_2$ has a finite subcover.

Suppose f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$. We first show that $f^{-1}(\mathcal{U}) = \{f^{-1}(U) | U \in \mathcal{U}\}$ is an open interval cover of $\mathcal{S}_1$. Since f^{-1} is an isomorphism and U is an open interval in $\mathcal{S}_2$, by Theorem 6.4, $f^{-1}(U)$ is an open interval in $\mathcal{S}_1$. Let $I \in \mathcal{S}_1$. Then, $f(I) \in \mathcal{S}_2$. Since $\mathcal{U}$ is an open interval cover of $\mathcal{S}_2$, there is $U_0 \in \mathcal{U}$ such that $f(I) \in U_0$, which means that U_0 is an open neighborhood of $f(I)$ in $\mathcal{S}_2$. By Theorem 6.4, $f^{-1}(U_0)$ is an open neighborhood of $f^{-1}(f(I)) = I$ in $\mathcal{S}_1$, which means $I \in f^{-1}(U_0)$. Thus, $f^{-1}(\mathcal{U})$ is an open interval cover of $\mathcal{S}_1$.

Since $(\mathcal{S}_1, \mathcal{I}_1)$ is open compact, $f^{-1}(\mathcal{U})$ has a finite subcover $f^{-1}(U_1)$, $f^{-1}(U_2)$, ..., $f^{-1}(U_n)$. We show that $U_1, U_2, ..., U_n$ is an open interval cover of $\mathcal{S}_2$. Let $J \in \mathcal{S}_2$ then $f^{-1}(J) \in \mathcal{S}_1$. Since $f^{-1}(U_1), f^{-1}(U_2), ..., f^{-1}(U_n)$ is an open interval cover of $\mathcal{S}_1$, there is $f^{-1}(U_i)$ $(1 \leq i \leq n)$ such that $f^{-1}(J) \in f^{-1}(U_i)$, which means that $f^{-1}(U_i)$ is an open neighborhood of $f^{-1}(J)$ in $(\mathcal{S}_1, \mathcal{I}_1)$. By Theorem 6.4, $f(f^{-1}(U_i)) = U_i$ is a neighborhood of $f(f^{-1}(J)) = J$ in $(\mathcal{S}_2, \mathcal{I}_2)$ which means $J \in U_i$. Now, we know that $U_1, U_2, ..., U_n$ is really an open interval cover of $\mathcal{S}_2$. This means the open interval cover $\mathcal{U}$ of $\mathcal{S}_2$ has a finite subcover. □

7. Conclusions and Future Work

In this paper, we show an undesirable property of closed convergence of information nets. Then, we introduce open information intervals and open neighborhoods, and thus introduce open convergence of information nets. Open convergence avoids the undesirable property of closed convergence. We point out a limitation of open convergence: for open convergence, a Moore-Smith style convergence theory cannot be established for general informalogical spaces. A Moore-Smith style convergence theory can only be established for open normal informalogical spaces. We also introduce open compactness of informalogical spaces and prove that an informalogical space is open compact if and only if each information net in the informalogical space has a subnet that open converges. Furthermore, we prove that open limit uniqueness, open normality, open separatedness, open first countability and open compactness are all isomorphic invariants.

Due to the importance of open normal informalogical spaces, in our future work, we will investigate conditions that make an informalogical space an open normal informalogical space. We already know that a linear informalogical space is an open normal informalogical space from Theorem 2.1 in Section 2. However, linearality is a very strong requirement. We want to find weaker conditions that still lead to open normality.

An isomorphism is a transformation from one informalogical space to another, and isomorphic transformations preserve a lot of properties of an informalogical space. However, isomorphism is a very strong requirement in that it requires preservation of both the order of information in an informalogical space and the informalogy of the informalogical space. In our future work, we will investigate weaker transformations and what properties of an informalogical space are preserved under the weaker transformations.

In Section 2, an example informalogical space $(\mathcal{S}, \mathcal{I})$ is given to show that the expression of an open interval is not generally unique and that the intersection of two open intervals is not generally an open interval. There are not a lot of pieces of information in the informalogical space. In that case, it is easier to understand the structure of the informalogical space and relationship among the pieces of information in $\mathcal{S}$ if $\mathcal{S}$ is represented visually by a directed graph. In our future work, we will investigate use of graphs to represent $\mathcal{S}$, especially when there are not a lot of pieces of information in $\mathcal{S}$.

In our future work, we will generalize intervals, neighborhoods and interval covers. We will also introduce topologies in an informalogical space.

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