Informalogical Space: Decomposition, Compactness and Isomorphism

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Abstract

In a previous paper, we presented a meaning based information theory in which the core concept is an *informalogical space*. We introduced the concepts of an *information net* in an informalogical space, a piece of *limit information* of an information net, a *separated* informalogical space and a *first countable* informalogical space. We built a convergence theory of information nets and applied the convergence theory to information sequences.

In this paper, we will introduce the concepts of *decompositions* of information and informalogical spaces, and we will prove some theorems about decompositions. We will introduce the concept of a *compact* informalogical space and prove that an informalogical space is compact if and only if each information net in the informalogical space has a subnet that converges. We will introduce the concept of an *isomorphism* between two informalogical spaces and the concept of an *isomorphic invariant* which is a property of informalogical spaces that is preserved under isomorphisms. We will prove that separatedness, limit uniqueness, first countability, and compactness are all isomorphic invariants.

Key words: information, informalogy, informalogical space, decomposition, compact, isomorphism.

1. Introduction

We are in an era of information explosion, especially after the introduction of the Internet. To tackle the information explosion, various efforts have been

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made to expend the theoretical capacity and application areas of information theory. (See [4], [5], [11], [14], [18], [33] and [51].) Much work has also been done in the related fields of artificial intelligence and knowledge discovery. (See [6], [15], [27], [29], [30], [32], [34], [36], [37], [38], [41], [42], [45], [46], [49], [50] and [53].) In these fields, many contributions have been made to machine learning or computational learning. (See [2], [3], [7], [8], [9], [16], [22], [23], [24], [27], [28], [33], [36], [37], [38] and [39].) Another related and active field is soft computing. Instead of computing with numbers, much research has focused on tackling the problem of computing with words, meaning, inference and reasoning based on meaning. (See [1], [10], [12], [17], [19], [20], [21], [26], [31], [33], [35], [40], [43], [44], [47], [48], [52], [54], [55], [56], [57] and [58].)

In [13], we presented a theory of information that is based on meanings of information and relationships among information. There, we introduced some basic concepts and the core concept of our approach, *informalogical space*. We introduced the concept of an *information net* in an informalogical space and discussed *convergence* of information nets and a special case of information nets, information sequences. We also introduced the concepts of a *separated* informalogical space and a *first countable* informalogical space. In this paper, we continue the theoretical build up in informalogical spaces. Before introducing new concepts, we revisit some basic concepts that were introduced in [13].

The *contain* relation between two pieces of information, and the *union* and *intersection* operations on information were introduced in [13]. Basically, suppose I and J are two pieces of information. If I can be inferred from J, then information I is *contained* in information J, and we can also say that information J *contains* information I. This relation is represented as $I \preceq J$ or $J \succeq I$.

As for the union and intersection operations on a non-empty information set $\mathcal{A}$, basically, the union $\vee\mathcal{A}$ is the sum of all the pieces of information in the information set $\mathcal{A}$, and the intersection $\wedge\mathcal{A}$ is the common information that is contained in each piece of information in the information set $\mathcal{A}$.

Below is the core concept of our approach, *informalogical space*, which was introduced in [13].

Definition 1.1. ([13]) *Let $\mathcal{S}$ be a non-empty information set, and let $\Omega = \vee\mathcal{S}$ (i.e., Ω is the union of all the information in $\mathcal{S}$). Let $\mathcal{I}$ be a non-empty subset of $\mathcal{S}$ such that $\vee\mathcal{I} = \Omega$. We say that $\mathcal{I}$ is an informalogy, that $\mathcal{S}$ is the space of the informalogy $\mathcal{I}$, that $\mathcal{I}$ is an informalogy for the space $\mathcal{S}$ and* *that the pair* $(\mathcal{S}, \mathcal{I})$ *is an informalogical space, if the following two conditions hold:*

1. *if* $I, J \in \mathcal{I}$, *then* $I \wedge J \in \mathcal{I}$; *and*
2. *if* $\mathcal{I}_0 \subseteq \mathcal{I}$, *then* $\vee \mathcal{I}_0 \in \mathcal{I}$.

After revisit of the above basic concepts in [13], we will present two theorems that contain some basic properties of information and information sets. As the same as assumed in [13], all the pieces of information under discussion are consistent information.

Theorem 1.1. *Suppose that* A, B, X *and* Y *are pieces of information, and* 0 *is the zero information. Then,*

1. $A \preceq A$;
2. *if* $X \preceq A$ *and* $A \preceq Y$, *then* $X \preceq Y$;
3. *if* $A \preceq X$ *and* $B \preceq X$, *then* $A \vee B \preceq X$;
4. *if* $X \preceq A$ *and* $X \preceq B$, *then* $X \preceq A \wedge B$;
5. $A \wedge B \preceq A \vee B$;
6. $A \vee B = B \vee A$, $A \wedge B = B \wedge A$;
7. *if* $X \preceq A$ *and* $Y \preceq B$, *then* $X \vee Y \preceq A \vee B$ *and* $X \wedge Y \preceq A \wedge B$;
8. $X \vee (A \vee B) = (X \vee A) \vee B$, $X \wedge (A \wedge B) = (X \wedge A) \wedge B$; *and*
9. $A \preceq B$, $A \vee B = B$ *and* $A \wedge B = A$ *are equivalent.*

PROOF. Items 1 and 2 come from Theorem 2.1 of [13]. Items 3 and 4 come from Theorem 2.2 of [13]. Items 5, and 6 are obvious from the definitions of union and intersection (see [13].)

7. $X \preceq A \preceq A \vee B$ and $Y \preceq B \preceq A \vee B$. By 3, $X \vee Y \preceq A \vee B$. $X \wedge Y \preceq X \preceq A$ and $X \wedge Y \preceq Y \preceq B$. By 4, $X \wedge Y \preceq A \wedge B$.

8. To prove $X \vee (A \vee B) = (X \vee A) \vee B$, we show $X \vee (A \vee B) \preceq (X \vee A) \vee B$ and $(X \vee A) \vee B \preceq X \vee (A \vee B)$. $A \preceq X \vee A \preceq (X \vee A) \vee B$. $B \preceq (X \vee A) \vee B$. By 3, $A \vee B \preceq (X \vee A) \vee B$. $X \preceq X \vee A \preceq (X \vee A) \vee B$. Again, by 3, we have $X \vee (A \vee B) \preceq (X \vee A) \vee B$. On the other hand, $X \preceq X \vee (A \vee B)$ and $A \preceq A \vee B \preceq X \vee (A \vee B)$. Thus, by 3, $X \vee A \preceq X \vee (A \vee B)$. $B \preceq A \vee B \preceq X \vee (A \vee B)$. Again, by 3, $(X \vee A) \vee B \preceq X \vee (A \vee B)$.

To prove $X \wedge (A \wedge B) = (X \wedge A) \wedge B$, we show $X \wedge (A \wedge B) \preceq (X \wedge A) \wedge B$ and $(X \wedge A) \wedge B \preceq X \wedge (A \wedge B)$. $X \wedge (A \wedge B) \preceq X$ and $X \wedge (A \wedge B) \preceq A \wedge B \preceq A$. By 4, $X \wedge (A \wedge B) \preceq X \wedge A$. $X \wedge (A \wedge B) \preceq A \wedge B \preceq B$. Again, by 4,

$X \wedge (A \wedge B) \preceq (X \wedge A) \wedge B$. On the other hand, $(X \wedge A) \wedge B \preceq X \wedge A \preceq A$ and $(X \wedge A) \wedge B \preceq B$, and thus by 4, $(X \wedge A) \wedge B \preceq A \wedge B$. $(X \wedge A) \wedge B \preceq X \wedge A \preceq X$. Again, by 4, $(X \wedge A) \wedge B \preceq X \wedge (A \wedge B)$.

9. Assume $A \preceq B$. We prove $A \vee B = B$ and $A \wedge B = A$. $A \preceq B$ and $B \preceq B$. By 3, $A \vee B \preceq B$. On the other hand, $B \preceq A \vee B$. Thus, $A \vee B = B$. $A \wedge B \preceq A$. On the other hand, $A \preceq A$ and $A \preceq B$. By 4, $A \preceq A \wedge B$. Thus, $A \wedge B = A$.

Assume $A \vee B = B$. Then, $A \preceq A \vee B = B$.

Assume $A \wedge B = A$. Then, $A = A \wedge B \preceq B$.

Now, we know that $A \preceq B$, $A \vee B = B$ and $A \wedge B = A$ are equivalent. □

Theorem 1.2. *Suppose that X, A and B are pieces of information, and $\mathcal{A}$ and $\mathcal{B}$ are information sets. Then,*

1. *if $A \preceq X$ for each $A \in \mathcal{A}$, then $\vee\mathcal{A} \preceq X$; if $X \preceq A$ for each $A \in \mathcal{A}$, then $X \preceq \wedge\mathcal{A}$;*
2. *if for each $A \in \mathcal{A}$, there is $B \in \mathcal{B}$ such that $A \preceq B$, then $\vee\mathcal{A} \preceq \vee\mathcal{B}$;*
3. *$X \vee (\vee\mathcal{A}) = \vee\{X \vee A | A \in \mathcal{A}\}$, $X \wedge (\wedge\mathcal{A}) = \wedge\{X \wedge A | A \in \mathcal{A}\}$; and*
4. *$(\vee\mathcal{A}) \vee (\vee\mathcal{B}) = \vee\{A \vee B | A \in \mathcal{A}, B \in \mathcal{B}\}$, $(\wedge\mathcal{A}) \wedge (\wedge\mathcal{B}) = \wedge\{A \wedge B | A \in \mathcal{A}, B \in \mathcal{B}\}$.*

Proof. Item 1 is obvious from the definitions of union and intersection (see [13].)

2. Since for each $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \preceq B$, $A \preceq \vee\mathcal{B}$. Therefore, $\vee\mathcal{A} \preceq \vee\mathcal{B}$.

3. To prove $X \vee (\vee\mathcal{A}) = \vee\{X \vee A | A \in \mathcal{A}\}$, we show $X \vee (\vee\mathcal{A}) \preceq \vee\{X \vee A | A \in \mathcal{A}\}$ and $\vee\{X \vee A | A \in \mathcal{A}\} \preceq X \vee (\vee\mathcal{A})$. $X \preceq X \vee A_0$ for any $A_0 \in \mathcal{A}$. Thus, $X \preceq \vee\{X \vee A | A \in \mathcal{A}\}$. For each $A \in \mathcal{A}$, $A \preceq X \vee A$. By 2, $\vee\mathcal{A} \preceq \vee\{X \vee A | A \in \mathcal{A}\}$. Thus by 3 of Theorem 1.1, $X \vee (\vee\mathcal{A}) \preceq \vee\{X \vee A | A \in \mathcal{A}\}$. On the other hand, for each $A \in \mathcal{A}$, $X \vee A \preceq X \vee (\vee\mathcal{A})$ since $A \preceq \vee\mathcal{A}$. Thus, by 1, $\vee\{X \vee A | A \in \mathcal{A}\} \preceq X \vee (\vee\mathcal{A})$.

To prove $X \wedge (\wedge\mathcal{A}) = \wedge\{X \wedge A | A \in \mathcal{A}\}$, we show $X \wedge (\wedge\mathcal{A}) \preceq \wedge\{X \wedge A | A \in \mathcal{A}\}$ and $\wedge\{X \wedge A | A \in \mathcal{A}\} \preceq X \wedge (\wedge\mathcal{A})$. For each $A \in \mathcal{A}$, $X \wedge (\wedge\mathcal{A}) \preceq X \wedge A$ since $\wedge\mathcal{A} \preceq A$. By 1, $X \wedge (\wedge\mathcal{A}) \preceq \wedge\{X \wedge A | A \in \mathcal{A}\}$. On the other hand, for a $A' \in \mathcal{A}$, $\wedge\{X \wedge A | A \in \mathcal{A}\} \preceq X \wedge A'$. Thus, $\wedge\{X \wedge A | A \in \mathcal{A}\} \preceq X$. For each $A_0 \in \mathcal{A}$, $\wedge\{X \wedge A | A \in \mathcal{A}\} \preceq X \wedge A_0 \preceq A_0$. By 1, $\wedge\{X \wedge A | A \in \mathcal{A}\} \preceq \wedge\mathcal{A}$. By 4 of Theorem 1.1, $\wedge\{X \wedge A | A \in \mathcal{A}\} \preceq X \wedge (\wedge\mathcal{A})$.

4. Using the first formula of 3, we have $(\vee\mathcal{A}) \vee (\vee\mathcal{B}) = \vee\{(\vee\mathcal{A}) \vee B | B \in \mathcal{B}\} = \vee\{B \vee (\vee\mathcal{A}) | B \in \mathcal{B}\} = \vee\{\vee\{B \vee A | A \in \mathcal{A}\} | B \in \mathcal{B}\} = \vee\{B \vee$

$A|A \in \mathcal{A}, B \in \mathcal{B}\} = \vee\{A \vee B|A \in \mathcal{A}, B \in \mathcal{B}\}$. Similarly, we can obtain $(\wedge\mathcal{A}) \wedge (\wedge\mathcal{B}) = \wedge\{A \wedge B|A \in \mathcal{A}, B \in \mathcal{B}\}$ by using the second formula of 3. □

The above is the revisit of some basic concepts that were introduced in [13] and two theorems that contain some basic properties of the union and intersection operations on information.

In Section 2 we will introduce the concepts of decomposition of information and decomposition of informalogies. We will also introduce the concept of base for an informalogy and prove a theorem that establishes the relationship of decomposition of a base and decomposition of an informalogy.

In Section 3 we will introduce the concept of a compact informalogical space. We will prove that an informalogical space is compact if and only if each information net in the informalogical space has a piece of cluster information, which also means that an informalogical space is compact if and only if each information net in the informalogical space has a subnet that converges to a piece of information in the space.

In Section 4 we will we consider functions (or, maps) between two informalogical spaces. We will introduce the concept of an isomorphism between two informalogical spaces and prove some theorems about isomorphisms.

In Section 5 we will introduce the concept of an isomorphic invariant which is a property of informalogical spaces that is preserved under isomorphisms. We will prove that separatedness, limit uniqueness, first countability and compactness are all isomorphic invariants.

In Section 6, we will conclude and briefly discuss some of our future work.

2. Decompositions of Information and Informalogical Spaces

In this section, we will introduce the concept of decomposition of information and informalogies. Decompositions will play an important role in transferring study of information under consideration to study of more basic pieces of information. We will also introduce the concept of base for an informalogy. First, we present a theorem that is useful later in this section.

Theorem 2.1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two information sets.*

1. *If $\mathcal{A} \subseteq \mathcal{B}$, then $\vee\mathcal{A} \preceq \vee\mathcal{B}$ and $\wedge\mathcal{B} \preceq \wedge\mathcal{A}$.*
2. *If for each $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ such that $B \preceq A$, then $\vee\mathcal{B} \preceq \vee\mathcal{A}$.*
3. *If for each $A \in \mathcal{A}$ there exists a subset $\mathcal{B}_A$ of $\mathcal{B}$ (i.e, $\mathcal{B}_A \subseteq \mathcal{B}$) such that $A = \vee\mathcal{B}_A$, then $\vee\mathcal{A} \preceq \vee\mathcal{B}$.*

4. *If for each* $A \in \mathcal{A}$ *there exists a subset* $\mathcal{B}_A$ *of* $\mathcal{B}$ *such that* $A = \vee\mathcal{B}_A$ *and* $\mathcal{B} = \cup\{\mathcal{B}_A | A \in \mathcal{A}\}$, *then* $\vee\mathcal{A} = \vee\mathcal{B}$.

PROOF. 1 and 2 are obvious from the definitions of union and intersection of information.

3. For each $A \in \mathcal{A}$, since $\mathcal{B}_A \subseteq \mathcal{B}$, by 1, $\vee\mathcal{B}_A \preceq \vee\mathcal{B}$. Thus, for each $A \in \mathcal{A}$, $A = \vee\mathcal{B}_A \preceq \vee\mathcal{B}$. Therefore, $\vee\mathcal{A} \preceq \vee\mathcal{B}$.

4. By 3, $\vee\mathcal{A} \preceq \vee\mathcal{B}$. On the other hand, for each $B \in \mathcal{B}$, there is $A \in \mathcal{A}$ such that $B \in \mathcal{B}_A$ since $\mathcal{B} = \cup\{\mathcal{B}_A | A \in \mathcal{A}\}$. Thus, $B \preceq \vee\mathcal{B}_A = A$. Then, by 2, $\vee\mathcal{B} \preceq \vee\mathcal{A}$. Now we know $\vee\mathcal{A} = \vee\mathcal{B}$. □

Now, we introduce the concept of decomposition.

Definition 2.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two non-empty information sets. We say that* $\mathcal{B}$ *is a decomposition information set of* $\mathcal{A}$ *if the following two conditions hold:*

1. *for each* $B \in \mathcal{B}$, *there is* $A_B \in \mathcal{A}$ *such that* $B \preceq A_B$; *and*
2. *for each* $A \in \mathcal{A}$, *there is a subset* $\mathcal{B}_A$ *of* $\mathcal{B}$ *(i.e.,* $\mathcal{B}_A \subseteq \mathcal{B}$*) such that* $A = \vee\mathcal{B}_A$.

When no confusions would arise, for short, we simply say that $\mathcal{B}$ *is a decomposition of* $\mathcal{A}$.

It's obvious that any information set is a decomposition of itself. This decomposition is a trivial decomposition.

By 2 and 3 of Theorem 2.1, $\vee\mathcal{B} = \vee\mathcal{A}$ if $\mathcal{B}$ is a decomposition of $\mathcal{A}$. If the information set $\mathcal{A}$ contains only a single piece of information, then the information set $\mathcal{B}$ is simply a decomposition of that piece of information. In essence, decomposition is to divide a piece of information into smaller and more basic pieces of information, but keep the total information unchanged. After decomposition, some pieces of information in the original information set may have common basic elements in the decomposition. Also, the study of the original pieces of information might be transferred into the study of the basic elements in the decomposition.

Informalogies in informalogical spaces are information sets. Thus, we can discuss decompositions of informalogies.

Definition 2.2. *Let* $(\mathcal{S}, \mathcal{I}_1)$ *and* $(\mathcal{S}, \mathcal{I}_2)$ *be two informalogical spaces. We say that informalogy* $\mathcal{I}_2$ *is a decomposition informalogy of informalogy* $\mathcal{I}_1$ *if, as two information sets,* $\mathcal{I}_2$ *is a decomposition of* $\mathcal{I}_1$*. When no confusions would arise, for short, we simply say that informalogy* $\mathcal{I}_2$ *is a decomposition of informalogy* $\mathcal{I}_1$

Definition 2.3. *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space, and let* $\mathcal{B}$ *be a subset of* $\mathcal{I}$ *(i.e.,* $\mathcal{B} \subseteq \mathcal{I}$*). We say that* $\mathcal{B}$ *is a base for the informalogy* $\mathcal{I}$ *if for each* $I \in \mathcal{I}$ *there is* $\mathcal{B}_I \subseteq \mathcal{B}$ *such that* $I = \vee\mathcal{B}_I$*, or in other words,* $\mathcal{B}$ *is a decomposition of* $\mathcal{I}$*.*

If we have a base for an informalogy, then the informalogy can be constructed by forming the unions of all the subsets of the base. By the way, an informalogy is a base for itself. This is a trivial base.

Theorem 2.2. *Let* $(\mathcal{S}, \mathcal{I}_1)$ *and* $(\mathcal{S}, \mathcal{I}_2)$ *be two informalogical spaces, and let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be two bases for the two informalogies* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *respectively. If, as two information sets,* $\mathcal{B}_2$ *is a decomposition of* $\mathcal{B}_1$*, then, informalogy* $\mathcal{I}_2$ *is a decomposition of informalogy* $\mathcal{I}_1$*.*

PROOF. First, we show that for each $J \in \mathcal{I}_2$, there exists $I_J \in \mathcal{I}_1$ such that $J \preceq I_J$. Since $\mathcal{B}_2$ is a base for $\mathcal{I}_2$, there exists a subset $\mathcal{B}_{2J}$ of $\mathcal{B}_2$ such that $J = \vee\mathcal{B}_{2J}$. Since $\mathcal{B}_2$ is a decomposition of $\mathcal{B}_1$, for each $K \in \mathcal{B}_{2J}$, there is $L_K \in \mathcal{B}_1$ such that $K \preceq L_K$. Then, $I_J \equiv \vee\{L_K | K \in \mathcal{B}_{2J}\} \in \mathcal{I}_1$, and by 2 of Theorem 2.1, $J = \vee\mathcal{B}_{2J} \preceq \vee\{L_K | K \in \mathcal{B}_{2J}\} = I_J$.

Next, we show that for each $I \in \mathcal{I}_1$ there exists a subset $\mathcal{I}_{2I}$ of $\mathcal{I}_2$ such that $I = \vee\mathcal{I}_{2I}$. Since $\mathcal{B}_1$ is a base for $\mathcal{I}_1$, there exists a subset $\mathcal{B}_{1I}$ of $\mathcal{B}_1$ such that $I = \vee\mathcal{B}_{1I}$. Since $\mathcal{B}_2$ is a decomposition of $\mathcal{B}_1$, for each $L \in \mathcal{B}_{1I}$ there exists a subset $\mathcal{B}_{2L}$ of $\mathcal{B}_2$ such that $L = \vee\mathcal{B}_{2L}$. Then, $\mathcal{I}_{2I} \equiv \cup\{\mathcal{B}_{2L} | L \in \mathcal{B}_{1I}\} \subseteq \mathcal{B}_2 \subseteq \mathcal{I}_2$, and by 4 of Theorem 2.1, $I = \vee\mathcal{B}_{1I} = \vee\mathcal{I}_{2I}$.

Now, we know that informalogy $\mathcal{I}_2$ is a decomposition of informalogy $\mathcal{I}_1$.

□

What Theorem 2.2 tells us is that decomposition relationship between the bases for two informalogies implies decomposition relationship between the two informalogies. Thus, judging whether one informalogy is a decomposition of another informalogy may be reduced to checking whether a base for the first informalogy is a decomposition of a base for the second informalogy. It should be noted that generally the decomposition relationship between two informalogies does not imply decomposition relationship between arbitrary bases for the two informalogies.

3. Compactness

In [13], we introduced the concept of *information interval* (or, *interval*, for short) in an informalogical space. In this paper, based on the concept of interval, we introduce the concept of *interval cover* of an information set in an informalogical space, and based on the concept of interval cover we introduce the concept of a *compact* informalogical space. First, we revisit the definitions of information interval in an informalogical space and neighborhood of a piece of information in an informalogical space.

Definition 3.1. ([13]) *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space. Let* X *and* Y *be two members of the informalogy* $\mathcal{I}$*. We define* $[X, Y]$ *as* $[X, Y] \equiv \{I | I \in \mathcal{S} \text{ and } X \preceq I \preceq Y\}$*.* $[X, Y]$ *is an information set which contains all the information in* $\mathcal{S}$ *that ranges from the lower endpoint* X *to the upper endpoint* Y*. We call* $[X, Y]$ *an information interval in the informalogical space* $(\mathcal{S}, \mathcal{I})$*, or simply an interval. When* $[X, Y]$ *is non-empty, we call it a non-empty interval; when* $[X, Y]$ *is empty, we call it an empty interval, and we use* θ *to denote an empty interval.*

When both $[X_1, Y_1]$ *and* $[X_2, Y_2]$ *are information intervals, and* $[X_1, Y_1] \subseteq [X_2, Y_2]$*, we say that* $[X_1, Y_1]$ *is a subinterval of* $[X_2, Y_2]$*.*

In cases where no confusion is likely to result, we may simply use a single letter such as U*,* V*, etc. to represent an information interval.*

A set of intervals is called a family of intervals, or an interval family. We often use $\mathcal{U}$ *to represent an interval family.*

Definition 3.2. ([13]) *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space, and let* $I \in \mathcal{S}$*. Let* $[X, Y]$ *be a non-empty interval in the informalogical space. If* $I \in [X, Y]$*, which means* $X \preceq I \preceq Y$*, we say that the interval* $[X, Y]$ *is an* $\mathcal{I}$*-neighborhood, or neighborhood for short, of* I*, and we use* $U_{(I)}[X, Y]$ *to denote this relationship. We can simply use* $[X, Y]$*,* $U_{(I)}$ *or* U *to denote a neighborhood if no confusion seems possible.*

With the definition of interval introduced, we can introduce the concept of an *interval cover* for a set of information in an informalogical space, and then we introduce the concept of a *compact* informalogical space using the concept of interval cover.

Definition 3.3. *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space, let* $\mathcal{A}$ *be a set of information in the informalogical space (i.e.,* $\mathcal{A} \subseteq \mathcal{S}$*), and let* $\mathcal{U}$ *be a family of* *intervals in the informalogical space. We say that the interval family* $\mathcal{U}$ *is an interval cover of the information set* $\mathcal{A}$ *if* $\mathcal{A} \subseteq \cup\mathcal{U}$*, or in other words, each piece of information in* $\mathcal{A}$ *is in some interval in the interval family* $\mathcal{U}$*.*

Definition 3.4. *We say that an informalogical space* $(\mathcal{S}, \mathcal{I})$ *is a compact informalogical space, or* $(\mathcal{S}, \mathcal{I})$ *is compact, if each interval cover of the space* $\mathcal{S}$ *has a finite subcover.*

In other words, when $(\mathcal{S}, \mathcal{I})$ is compact, then, for any interval cover $\mathcal{U}$ of the space $\mathcal{S}$, there are finite intervals U_1, U_2, ..., U_n in the interval family $\mathcal{U}$ such that $\mathcal{A} \subseteq \cup_{i=1}^{n} U_i$.

A compact informalogical space has some good properties. In [13], we introduced information nets and built a convergence theory of information nets which includes the concepts and theorems about limit information and cluster information. Below, we prove that an informalogical space is compact if and only if each information net in the informalogical space has a piece of cluster information. First, we revisit the concepts related to information nets in an informalogical space and their convergence.

Definition 3.5. ([25, Chapter 2]) *We say that a binary relation* $\geq$ *directs a set* D*, and that the pair* $(D, \geq)$ *is a directed set, if* D *is non-empty, and*

1. *if* m*,* n *and* p *are members of* D *such that* $m \geq n$ *and* $n \geq p$*, then* $m \geq p$*;*
2. *if* $m \in D$*, then* $m \geq m$*; and*
3. *if* m *and* n *are members of* D*, then there is* p *in* D *such that* $p \geq m$ *and* $p \geq n$*.*

Definition 3.6. ([13]) *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space,* $(D, \geq)$ *be a directed set and* T *be a function on* D *whose values are pieces of information in the space. That means, for each* $n \in D$*, there is one and only one* $T_n \in \mathcal{S}$ *that corresponds to* n*. Then,* $\{T_n, n \in D, \geq\}$ *is called an information net in the space* $\mathcal{S}$*. In cases where no confusion would result, we simply use* $\{T_n, n \in D\}$ *or* $\{T_n\}$ *to denote an information net.*

Definition 3.7. ([13]) *Let* $\{T_n, n \in D, \geq\}$ *be an information net, and let* $[X, Y]$ *be an information interval. Then*

1. *we say that the information net* $\{T_n, n \in D, \geq\}$ *is in the information interval* $[X, Y]$ *if* $T_n \in [X, Y]$ *for every* $n \in D$*;*

2. *we say that the information net* $\{T_n, n \in D, \geq\}$ *is eventually in the information interval* $[X, Y]$ *if there is an* $m \in D$ *such that* $T_n \in [X, Y]$ *for every* $n \in D$ *that satisfies* $n \geq m$*; and*
3. *we say that the information net* $\{T_n, n \in D, \geq\}$ *is frequently in the information interval* $[X, Y]$ *if, for every* $m \in D$*, there is* $n \in D$ *such that* $n \geq m$ *and* $T_n \in [X, Y]$*.*

Definition 3.8. ([13]) *Let* $(\mathcal{S}, \mathcal{I})$ *be an informalogical space,* $\{T_n, n \in D, \geq\}$ *be an information net in the space, and* I *be a piece of information in the space. We say that the information net* $\{T_n, n \in D, \geq\}$ *converges to the information* I *in the informalogical space* $(\mathcal{S}, \mathcal{I})$*, or say that* $\{T_n, n \in D, \geq\}$ $\mathcal{I}$*-converges to* I*, if the information net* $\{T_n, n \in D, \geq\}$ *is eventually in every neighborhood of* I*. The information* I *is called a piece of* $\mathcal{I}$*-limit information of the information net* $\{T_n, n \in D, \geq\}$ *if* $\{T_n, n \in D, \geq\}$ $\mathcal{I}$*-converges to* I*. When no confusion would arise, for short, we simply say that the information net* $\{T_n, n \in D, \geq\}$ *converges to information* I*, and that the information* I *is a piece of limit information of the information net* $\{T_n, n \in D, \geq\}$*.*

Definition 3.9. ([13]) *We say that a piece of information* I *is a piece of cluster information of an information net* $\{T_n\}$ *if the information net* $\{T_n\}$ *is frequently in every neighborhood of* I*.*

Now, we prove the following theorem that establish the equivalent relationship between a compact informalogical space and the property that each information net in the informalogical space has a subnet that converges. We achieve this by first proving the following theorem about compactness and an information net having a piece of cluster information.

Theorem 3.1. *An informalogical space* $(\mathcal{S}, \mathcal{I})$ *is compact if and only if each information net in the informalogical space has a piece of cluster information.*

Proof. Suppose that $(\mathcal{S}, \mathcal{I})$ is a compact informalogical space, and suppose that $\{T_n, n \in D, \geq\}$ is an information net in the informalogical space. If the information net $\{T_n, n \in D, \geq\}$ has no cluster information, then, for each piece of information I in the space $\mathcal{S}$ (*i.e.*, $I \in \mathcal{S}$), there is a neighborhood U_I of I (*i.e.*, U_I is an information interval and $I \in U_I$) and $n_I \in D$ such that $T_n \notin U_I$ for $n \in D$ and $n \geq n_I$. Let $\mathcal{U} \equiv \{U_I | I \in \mathcal{S}\}$, then $\mathcal{U}$ is an interval cover of $\mathcal{S}$. Since $(\mathcal{S}, \mathcal{I})$ is compact, $\mathcal{U}$ has a finite subcover $\{U_{I1}, U_{I2}, ..., U_{Ir}\}$.

Since D is a directed set, there is $n_0 \in D$ such that $n_0 \geq n_{Ij}$, $j = 1, 2, ..., r$. Therefore, $T_{n0} \notin U_{Ij}$, $j = 1, 2, ..., r$. However, this is contradictory to that $\{U_{I1}, U_{I2}, ..., U_{Ir}\}$ is an interval cover of the space $\mathcal{S}$. This proves that if $(\mathcal{S}, \mathcal{I})$ is compact then each information net in the informalogical space has a piece of cluster information.

Conversely, suppose that each information net in $(\mathcal{S}, \mathcal{I})$ has a piece of cluster information, and suppose that $\mathcal{U}$ is an interval cover of $\mathcal{S}$. We will show that $\mathcal{U}$ has a finite subcover. Otherwise, for each finite subfamily $n \equiv \{U_1, U_2, ..., U_r\}$ of $\mathcal{U}$ (*i.e.*, $U_j \in \mathcal{U}, j = 1, 2, ..., r$), there is $T_n \in \mathcal{S}$ such that $T_n \notin U_j, j = 1, 2, ..., r$. Let $D \equiv \{n | n \text{ is a finite subfamily of } \mathcal{U}\}$. We will show that $(D, \supseteq)$ is a directed set. (Here, $\supseteq$ is the regular superset relation between two sets.) For $n_1, n_2, n_3 \in D$, if $n_1 \supseteq n_2$ and $n_2 \supseteq n_3$, then $n_1 \supseteq n_3$. A set n is a superset of itself, meaning $n \supseteq n$. For any two members $n_1, n_2 \in D$, let $n = n_1 \cup n_2$. Then, since n_1 and n_2 are two finite subfamilies of $\mathcal{U}$, n is also a finite subfamily of $\mathcal{U}$ and thus $n \in D$. It's obvious that $n \supseteq n_1$ and $n \supseteq n_2$. Now, we know that $(D, \supseteq)$ is a directed set. Therefore, $\{T_n, n \in D, \supseteq\}$ is an information net in the informalogical space $(\mathcal{S}, \mathcal{I})$.

Since each information net in $(\mathcal{S}, \mathcal{I})$ has a piece of cluster information, let T be a piece of cluster information of the information net $\{T_n, n \in D, \supseteq\}$. Since $\mathcal{U}$ is an interval cover of $\mathcal{S}$, there is $U_T \in \mathcal{U}$ such that $T \in U_T$, which means that U_T is a neighborhood of T. Let $n_0 \equiv \{U_T\} \in D$. Then, since T is a piece of cluster information of $\{T_n, n \in D, \supseteq\}$, there is an $n \in D$ such that $n \supseteq n_0$ and T_n is in the neighborhood U_T of T, or in other words, $T_n \in U_T$. However, on the other hand, $n \supseteq n_0$ means $n \supseteq \{U_T\}$ which implies $U_T \in n$. Thus, by the choice of T_n, T_n is not in any interval in n, which implies $T_n \notin U_T$. Now, we have $T_n \in U_T$ and $T_n \notin U_T$ at the same time. This contradiction shows that $\mathcal{U}$ must have a finite subcover. □

Theorem 5.1 of [13] shows that a piece of information is a piece of cluster information of an information net if and only if the information net has a subnet that converges to that piece of information. Thus, by the preceding theorem, we can have the following corollary.

Corollary 3.1. *An informalogical space $(\mathcal{S}, \mathcal{I})$ is compact if and only if each information net in the informalogical space has a subnet that converges to a piece of information in $(\mathcal{S}, \mathcal{I})$.*

4. Isomorphisms

In this section we consider functions between two informalogical spaces. A function from informalogical space $(\mathcal{S}_1, \mathcal{I}_1)$ to informalogical space $(\mathcal{S}_2, \mathcal{I}_2)$ is a function from the information set $\mathcal{S}_1$ to the information set $\mathcal{S}_2$.

Definition 4.1. *Let f be a function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. We say that f is an order-preserving function if $I \preceq J$ in $\mathcal{S}_1$ implies $f(I) \preceq f(J)$ in $\mathcal{S}_2$, and $f(I) \preceq f(J)$ in $\mathcal{S}_2$ implies $I \preceq J$ in $\mathcal{S}_1$.*

It's clear that an order-preserving function preserves the *contain* relationship between two pieces of information. It's easy to see that, for a bijective function f, f^{-1} is order-preserving if and only if f itself is order-preserving. (Here, f^{-1} is the inverse function of f.)

Theorem 4.1. *Let f be an order-preserving bijective function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, and let $\Omega_1 = \vee \mathcal{S}_1$ and $\Omega_2 = \vee \mathcal{S}_2$. Then, $f(0) = 0$ and $f(\Omega_1) = \Omega_2$.*

Proof. We know that in an informalogical space $(\mathcal{S}, \mathcal{I})$, the informalogy $\mathcal{I}$ (and thus the space $\mathcal{S}$) contains at least two pieces of information, the zero information 0 and the *reference information* $\Omega = \vee \mathcal{I} = \vee \mathcal{S}$ (See [13].) Thus, we know that $0, \Omega_1 \in \mathcal{I}_1 \subseteq \mathcal{S}_1$ and $0, \Omega_2 \in \mathcal{I}_2 \subseteq \mathcal{S}_2$. Since f is a bijective function from $\mathcal{S}_1$ to $\mathcal{S}_2$, there is $I \in \mathcal{S}_1$ such that $f(I) = 0$. Since $f(I) = 0 \preceq f(0)$ and f is an order-preserving function, $I \preceq 0$. Thus, $I = 0$, and therefore, $f(0) = 0$. Again, since f is a bijective function from $\mathcal{S}_1$ to $\mathcal{S}_2$, there is $J \in \mathcal{S}_1$ such that $f(J) = \Omega_2$. Since $f(\Omega_1) \preceq \Omega_2 = f(J)$, and f is an order-preserving function, $\Omega_1 \preceq J$. Thus, $J = \Omega_1$ and therefore $f(\Omega_1) = \Omega_2$. □

Theorem 4.2. *Let f be an order-preserving bijective function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Let $A, B \in \mathcal{S}_1$. Then, $f(\{I | I \in \mathcal{S}_1$ and $A \preceq I \preceq B\}) = \{J | J \in \mathcal{S}_2$ and $f(A) \preceq J \preceq f(B)\}$.*

Proof. Let $X \in f(\{I | I \in \mathcal{S}_1 \text{ and } A \preceq I \preceq B\})$. Then, there is $I \in \mathcal{S}_1$ satisfying $A \preceq I \preceq B$ such that $f(I) = X$. Since f is an order-preserving function, $f(A) \preceq f(I) \preceq f(B)$. This means $f(A) \preceq X \preceq f(B)$. Thus, $X \in \{J | J \in \mathcal{S}_2 \text{ and } f(A) \preceq J \preceq f(B)\}$.

On the other hand, let $X \in \{J | J \in \mathcal{S}_2 \text{ and } f(A) \preceq J \preceq f(B)\}$. Then, $f(A) \preceq X \preceq f(B)$. Since f is a bijective function, there is $I \in \mathcal{S}_1$ such that $f(I) = X$. Thus, $f(A) \preceq f(I) \preceq f(B)$. Since f is order-preserving, $A \preceq I \preceq B$. That is, $X \in f(\{I | I \in \mathcal{S}_1 \text{ and } A \preceq I \preceq B\})$.

Combination of the above two facts shows $f(\{I | I \in \mathcal{S}_1 \text{ and } A \preceq I \preceq B\}) = \{J | J \in \mathcal{S}_2 \text{ and } f(A) \preceq J \preceq f(B)\}$. □

Definition 4.2. *Let $(\mathcal{S}_1, \mathcal{I}_1)$ and $(\mathcal{S}_2, \mathcal{I}_2)$ be two informalogical spaces, let f be a bijective function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, and let f be an order-preserving function. We say that f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$ and that $(\mathcal{S}_1, \mathcal{I}_1)$ is isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$ if $I \in \mathcal{I}_1$ implies $f(I) \in \mathcal{I}_2$ and $f(I) \in \mathcal{I}_2$ implies $I \in \mathcal{I}_1$.*

It's obvious that f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$ if f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. It's also easy to verify that $g \bullet f$ is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_3, \mathcal{I}_3)$ if f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$ and g is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_3, \mathcal{I}_3)$. (Here, "$\bullet$" is the function composition.)

Theorem 4.3. *Let f be an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, and let $[X, Y]$ be an information interval in $(\mathcal{S}_1, \mathcal{I}_1)$. Then, $f([X, Y]) = [f(X), f(Y)]$ is an information interval in $(\mathcal{S}_2, \mathcal{I}_2)$.*

PROOF. $[X, Y]$ being an information interval in $(\mathcal{S}_1, \mathcal{I}_1)$ implies $X, Y \in \mathcal{I}_1$. Since f is an isomorphism, $f(X), f(Y) \in \mathcal{I}_2$ and thus, $[f(X), f(Y)]$ is an information interval in $(\mathcal{S}_2, \mathcal{I}_2)$. Since f is an order-preserving bijective function, by Theorem 4.2, $f([X, Y]) = [f(X), f(Y)]$. □

Below, we prove a theorem that shows the neighborhood system of a piece of information is preserved under an isomorphism.

Theorem 4.4. *Let f be an order-preserving bijective function from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then, the following two statements are equivalent.*

1. *f is an isomorphism.*
2. *$[f(X), f(Y)]$ is a neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if $[X, Y]$ is a neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$.*

PROOF. We know that f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$ if f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$.

First, suppose 1. is true. We prove 2. is true. Assume $[X, Y]$ is a neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$, then, $X, Y \in \mathcal{I}_1$ and $X \preceq I \preceq Y$. Since f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, $f(X), f(Y) \in \mathcal{I}_2$ and $f(X) \preceq f(I) \preceq f(Y)$. This means that $[f(X), f(Y)]$ is a neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Conversely, assume $[f(X), f(Y)]$ is a neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, then by what we just proved, $[f^{-1}(f(X)), f^{-1}(f(Y))] = [X, Y]$ is a neighborhood of $f^{-1}(f(I)) = I$ in $(\mathcal{S}_1, \mathcal{I}_1)$.

Second, suppose 2. is true. We prove 1. is true, which means we prove that $f(I) \in \mathcal{I}_2$ if and only if $I \in \mathcal{I}_1$. Assume $I \in \mathcal{I}_1$. Then, $[I, I]$ is a neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$. By 2., $[f(I), f(I)]$ is a neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$, which implies $f(I) \in \mathcal{I}_2$. Conversely, assume $f(I) \in \mathcal{I}_2$. Then, $[f(I), f(I)]$ is a neighborhood of $f(I)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. By 2., $[I, I]$ is a neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$, which implies $I \in \mathcal{I}_1$. □

With the establishment of the above theorem about the neighborhood system of a piece of information under an isomorphism, we will prove a theorem that shows the convergence of an information net is preserved under an isomorphism.

Let f be an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Let $\{T_n, n \in D, \geq\}$ be an information net in $(\mathcal{S}_1, \mathcal{I}_1)$. Then, $\{f(T_n), n \in D, \geq\}$ is an information net in $(\mathcal{S}_2, \mathcal{I}_2)$. The following theorem establishes the relationship between the convergence of the two information nets in the two informalogical spaces.

Theorem 4.5. *Let f be an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. The information net $\{f(T_n), n \in D, \geq\}$ converges to $f(T)$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if the information net $\{T_n, n \in D, \geq\}$ converges to T in $(\mathcal{S}_1, \mathcal{I}_1)$.*

Proof. Assume that $\{T_n, n \in D, \geq\}$ converges to T in $(\mathcal{S}_1, \mathcal{I}_1)$. We consider $\{f(T_n), n \in D, \geq\}$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Let $[f(X), f(Y)]$ be a neighborhood of $f(T)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. By Theorem 4.4, $[X, Y]$ is a neighborhood of T in $(\mathcal{S}_1, \mathcal{I}_1)$. Since $\{T_n, n \in D, \geq\}$ converges to T in $(\mathcal{S}_1, \mathcal{I}_1)$, there is $p \in D$ such that $T_n \in [X, Y]$ for $n \in D$ satisfying $n \geq p$. $T_n \in [X, Y]$ means that $[X, Y]$ is a neighborhood of T_n in $(\mathcal{S}_1, \mathcal{I}_1)$. By Theorem 4.4, $[f(X), f(Y)]$ is a neighborhood of $f(T_n)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. This means $f(T_n) \in [f(X), f(Y)]$ for $n \in D$ satisfying $n \geq p$. Thus, $\{f(T_n), n \in D, \geq\}$ converges to $f(T)$ in $(\mathcal{S}_2, \mathcal{I}_2)$.

Conversely, assume that the information net $\{f(T_n), n \in D, \geq\}$ converges to $f(T)$ in $(\mathcal{S}_2, \mathcal{I}_2)$. Then, since f^{-1} is an isomorphism from $(\mathcal{S}_2, \mathcal{I}_2)$ to $(\mathcal{S}_1, \mathcal{I}_1)$, by what we just proved, the information net $\{f^{-1}(f(T_n)), n \in D, \geq\} = \{T_n, n \in D, \geq\}$ converges to $f^{-1}(f(T)) = T$ in $(\mathcal{S}_1, \mathcal{I}_1)$. □

By the preceding theorem and Theorem 4.3 of [13] we can have the following corollary.

Corollary 4.1. *Let f be an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, let $\mathcal{A}$ be an information set in $(\mathcal{S}_1, \mathcal{I}_1)$ and let I be a piece of information in $(\mathcal{S}_1, \mathcal{I}_1)$. Then, $f(I)$ is an accumulation information of the information set $f(\mathcal{A})$ in $(\mathcal{S}_2, \mathcal{I}_2)$ if and only if I is an accumulation information of the information set $\mathcal{A}$ in $(\mathcal{S}_1, \mathcal{I}_1)$.*

5. Isomorphic Invariants

In this section we discuss properties of informalogical spaces that are preserved under isomorphisms.

Definition 5.1. *We say that a property of informalogical spaces is an isomorphic invariant if the property which when possessed by an informalogical space is also possessed by each isomorphic informalogical space.*

Paper [13] introduced the concept of a *separated* informalogical space and proved that, if exists, the limit information of an information net in a separated informalogical space is unique. Here, we prove that separatedness is an isomorphic invariant. First, we revisit the definition of a separated informalogical space. In the definition, θ is the empty information interval (see [13].)

Definition 5.2. ([13]) *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space. We say that $(\mathcal{S}, \mathcal{I})$ is a separated informalogical space, or say that it is separated, if for every two distinct pieces of information I and J in the space, i.e., $I, J \in \mathcal{S}$ and $I \neq J$, there exist neighborhoods U and V of I and J, respectively, such that $U \cap V = \theta$.*

Theorem 5.1. *Separatedness is an isomorphic invariant.*

Proof. Let $(\mathcal{S}_1, \mathcal{I}_1)$ be a separated informalogical space, and let $(\mathcal{S}_1, \mathcal{I}_1)$ be isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$. We will prove that $(\mathcal{S}_2, \mathcal{I}_2)$ is also separated.

Let J_1 and J_2 be two distinct pieces of information in $(\mathcal{S}_2, \mathcal{I}_2)$, or in other words, $J_1, J_2 \in \mathcal{S}_2$ and $J_1 \neq J_2$. Suppose f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then, $f^{-1}(J_1), f^{-1}(J_2) \in \mathcal{S}_1$ and $f^{-1}(J_1) \neq f^{-1}(J_2)$. Since $(\mathcal{S}_1, \mathcal{I}_1)$ is separated, there are neighborhoods $[X_1, Y_1]$ and $[X_2, Y_2]$ of $f^{-1}(J_1)$ and

$f^{-1}(J_2)$, respectively, in $(\mathcal{S}_1, \mathcal{I}_1)$ such that $[X_1, Y_1] \cap [X_2, Y_2] = \theta$. By Theorem 4.4, $[f(X_1), f(Y_1)]$ and $[f(X_2), f(Y_2)]$ are neighborhoods of $f(f^{-1}(J_1)) = J_1$ and $f(f^{-1}(J_2)) = J_2$, respectively, in $(\mathcal{S}_2, \mathcal{I}_2)$.

Next, we show that $[f(X_1), f(Y_1)] \cap [f(X_2), f(Y_2)] = \theta$. Otherwise, there is $J \in [f(X_1), f(Y_1)] \cap [f(X_2), f(Y_2)] \subseteq \mathcal{S}_2$. $J \in [f(X_1), f(Y_1)]$ implies $f(X_1) \preceq J \preceq f(Y_1)]$ and $J \in [f(X_2), f(Y_2)]$ implies $f(X_2) \preceq J \preceq f(Y_2)]$. Since f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, there is $I \in \mathcal{S}_1$ such that $f(I) = J$. Thus, $f(X_1) \preceq f(I) \preceq f(Y_1)$ and $f(X_2) \preceq f(I) \preceq f(Y_2)$. Again, since f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$, we have $X_1 \preceq I \preceq Y_1$ and $X_2 \preceq I \preceq Y_2$. These mean that $I \in [X_1, Y_1]$ and $I \in [X_2, Y_2]$. Thus, $I \in [X_1, Y_1] \cap [X_2, Y_2]$. However, this is contradictory to the fact that $[X_1, Y_1] \cap [X_2, Y_2] = \theta$. Thus, we must have $[f(X_1), f(Y_1)] \cap [f(X_2), f(Y_2)] = \theta$. This proves that $(\mathcal{S}_2, \mathcal{I}_2)$ is separated. □

Theorem 4.2 of [13] showed that an informalogical space is separated if and only if every information net in the informalogical space has at most one piece of limit information. Thus, by the preceding theorem, we have the following corollary.

Corollary 5.1. *Limit uniqueness is an isomorphic invariant.*

Paper [13] introduced the concept of a *first countable* informalogical space. Here, we prove that first countability is an isomorphic invariant. First, we revisit the definitions of neighborhood system, base for a neighborhood system, and first countable informalogical space from [13].

Definition 5.3. ([13]) *We say that the family of all neighborhoods of a piece of information I is the neighborhood system of I, and we often use $\mathcal{U}_I$ to denote the neighborhood system of I.*

If $\mathcal{U}_0 \subseteq \mathcal{U}_I$, and every neighborhood of I contains a member of $\mathcal{U}_0$ as a subinterval, we say that $\mathcal{U}_0$ is a base for the neighborhood system of I, or a local base at I.

Definition 5.4. ([13]) *Let $(\mathcal{S}, \mathcal{I})$ be an informalogical space. We say that the informalogical space is first countable if the neighborhood family of each piece of information in the space has a countable base. In other words, there is a countable local base at each piece of information in the space.*

Theorem 5.2. *First countability is an isomorphic invariant.*

PROOF. Let $(\mathcal{S}_1, \mathcal{I}_1)$ be a first countable informalogical space, and let $(\mathcal{S}_1, \mathcal{I}_1)$ be isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$. We will prove that $(\mathcal{S}_2, \mathcal{I}_2)$ is also first countable, or in other words, each piece of information J in $\mathcal{S}_2$ has a countable local base.

Suppose f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. Then, there is $I \in \mathcal{S}_1$ such that $f(I) = J$. Since $(\mathcal{S}_1, \mathcal{I}_1)$ is first countable, I has a countable local base $[X_1, Y_1], [X_2, Y_2], ..., [X_n, Y_n], ...$ We will show that $[f(X_1), f(Y_1)], [f(X_2), f(Y_2)], ..., [f(X_n), f(Y_n)], ...$ is a local base at J in $(\mathcal{S}_2, \mathcal{I}_2)$.

By Theorem 4.4, $[f(X_i), f(Y_i)]$ (i=1, 2, ..., n, ...) is a neighborhood of J. Thus, $[f(X_1), f(Y_1)], [f(X_2), f(Y_2)], ..., [f(X_n), f(Y_n)], ...$ is a subset of the neighborhood system of J. Let $[A, B]$ be a neighborhood of J. Then, by Theorem 4.4, $[f^{-1}(A), f^{-1}(B)]$ is a neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$. Since $[X_1, Y_1], [X_2, Y_2], ..., [X_n, Y_n], ...$ is a local base at I, there is $[X_i, Y_i]$ such that $[X_i, Y_i] \subseteq [f^{-1}(A), f^{-1}(B)]$. Thus, $f^{-1}(A) \preceq X_i \preceq Y_i \preceq f^{-1}(B)$. Since f is an isomorphism, $f(f^{-1}(A)) \preceq f(X_i) \preceq f(Y_i) \preceq f(f^{-1}(B))$ which is $A \preceq f(X_i) \preceq f(Y_i) \preceq B$. This means $[f(X_i), f(Y_i)] \subseteq [A, B]$. Now, we know that $[f(X_1), f(Y_1)], [f(X_2), f(Y_2)], ..., [f(X_n), f(Y_n)], ...$ is a local base at J. □

We just introduced the concept of a *compact* informalogical space in Section 3 of this paper. Now, we prove that compactness is an isomorphic invariant.

Theorem 5.3. *Compactness is an isomorphic invariant.*

PROOF. Let $(\mathcal{S}_1, \mathcal{I}_1)$ be a compact informalogical space, and let $(\mathcal{S}_1, \mathcal{I}_1)$ be isomorphic to $(\mathcal{S}_2, \mathcal{I}_2)$. We will prove that $(\mathcal{S}_2, \mathcal{I}_2)$ is also compact, or in other words, each interval cover $\mathcal{U}$ of $\mathcal{S}_2$ has a finite subcover.

Suppose f is an isomorphism from $(\mathcal{S}_1, \mathcal{I}_1)$ to $(\mathcal{S}_2, \mathcal{I}_2)$. We first show that $f^{-1}(\mathcal{U}) = \{f^{-1}([X, Y]) | [X, Y] \in \mathcal{U}\}$ is an interval cover of $\mathcal{S}_1$. By Theorem 4.3, $f^{-1}([X, Y]) = [f^{-1}(X), f^{-1}(Y)])$ is an interval in $\mathcal{S}_1$. We will show that $\{[f^{-1}(X), f^{-1}(Y)] | [X, Y] \in \mathcal{U}\}$ is an interval cover of $\mathcal{S}_1$. Let $I \in \mathcal{S}_1$. Then, $f(I) \in \mathcal{S}_2$. Since $\mathcal{U}$ is an interval cover of $\mathcal{S}_2$, there is $[X_0, Y_0] \in \mathcal{U}$ such that $f(I) \in [X_0, Y_0]$, which means that $[X_0, Y_0]$ is a neighborhood of $f(I)$ in $\mathcal{S}_2$. By Theorem 4.4, $[f^{-1}(X_0), f^{-1}(Y_0)]$ is a neighborhood of $f^{-1}(f(I)) = I$ in $\mathcal{S}_1$, which means $I \in [f^{-1}(X_0), f^{-1}(Y_0)]$. Thus, $\{[f^{-1}(X), f^{-1}(Y)] | [X, Y] \in \mathcal{U}\}$ is an interval cover of $\mathcal{S}_1$.

Since $(\mathcal{S}_1, \mathcal{I}_1)$ is compact, $\{[f^{-1}(X), f^{-1}(Y)] | [X, Y] \in \mathcal{U}\}$ has a finite subcover $[f^{-1}(X_1), f^{-1}(Y_1)]$, $[f^{-1}(X_2), f^{-1}(Y_2)]$, ..., $[f^{-1}(X_n), f^{-1}(Y_n)]$. We will show that $[X_1, Y_1]$, $[X_2, Y_2]$, ..., $[X_n, Y_n]$ is an interval cover of $\mathcal{S}_2$. Let $J \in \mathcal{S}_2$. Then, there is $I \in \mathcal{S}_1$ such that $f(I) = J$. Since $[f^{-1}(X_1), f^{-1}(Y_1)]$, $[f^{-1}(X_2), f^{-1}(Y_2)]$, ..., $[f^{-1}(X_n), f^{-1}(Y_n)]$ is an interval cover of $\mathcal{S}_1$, there is $[f^{-1}(X_i), f^{-1}(Y_i)]$ $(1 \leq i \leq n)$ such that $I \in [f^{-1}(X_i), f^{-1}(Y_i)]$, which means that $[f^{-1}(X_i), f^{-1}(Y_i)]$ is a neighborhood of I in $(\mathcal{S}_1, \mathcal{I}_1)$. By Theorem 4.4, $[f(f^{-1}(X_i)), f(f^{-1}(Y_i))] = [X_i, Y_i]$ is a neighborhood of $f(I) = J$ in $(\mathcal{S}_2, \mathcal{I}_2)$ which means $J \in [X_i, Y_i]$. Now, we know that $[X_1, Y_1]$, $[X_2, Y_2]$, ..., $[X_n, Y_n]$ is an interval cover of $\mathcal{S}_2$. This means the interval cover $\mathcal{U}$ of $\mathcal{S}_2$ has a finite subcover. □

6. Conclusions and Future Work

This paper introduces decompositions of information and informalogical spaces, and proves some theorems about decompositions. This paper also introduces compact informalogical spaces and proves that an informalogical space is compact if and only if each information net in the informalogical space has a subnet that converges. This paper further introduces isomorphisms between two informalogical spaces and isomorphic invariants which are properties of informalogical spaces that are preserved under isomorphisms. This paper proves that separatedness, limit uniqueness, first countability, and compactness are all isomorphic invariants.

In our future work, we will investigate decompositions of information and informalogical spaces. Decompositions of information will play an important role in transferring study of information of interest to study of more basic pieces of information. The decompositions of an informalogical space $(\mathcal{S}, \mathcal{I})$ introduced in this paper only decompose the informalogy $\mathcal{I}$ but keep the space $\mathcal{S}$ the same. In our future work, we will investigate more general decompositions that not only decompose informalogies but also decompose spaces. We will also investigate *subbase* that can generate a base for an informalogy that is introduced in this paper.

Up to now, the neighborhood system of a piece of information in an informalogical space $(\mathcal{S}, \mathcal{I})$ is built with *closed* information intervals (*i.e.*, intervals containing two pieces of endpoint information), and the convergence theory of information nets is thus built with closed information intervals. In our future work, we will investigate *open* information intervals, neighborhood system built with open information intervals, and convergence theory of information nets built with open information intervals.

In our future work, we will also investigate functions between two informalogical spaces that possess only part the properties of an isomorphism between two informalogical spaces. Those functions will be more general.

References

[1] R. Alieva, A. Tserkovny, Systemic approach to fuzzy logic formalization for approximate reasoning, Information Sciences 181 (2011) 1045–1059

[2] E. Alpaydin, Introduction to Machine Learning (Adaptive Computation and Machine Learning), MIT Press, Cambridge, 2004.

[3] D. Angluin, Computational learning theory: Survey and selected bibliography, in: ACM Special Interest Group for Automata (Ed.), Proceedings of the Twenty-Fourth Annual ACM Symposium on Theory of Computing, ACM, Victoria, 1992, pp. 351–369.

[4] C. Arndt, Information Measures, Information and its Description in Science and Engineering, Springer-Verlag, New York, 2004.

[5] R.B. Ash, Information Theory, Dover, New York, 1990.

[6] E. Bartl, R. Belohlavek, Knowledge spaces with graded knowledge states, Information Sciences 181 (2011) 1426–1439

[7] L. Benuskova, N. Kasabov, Computational Neurogenetic Modeling, Springer, New York, 2007.

[8] C.M. Bishop, Pattern Recognition and Machine Learning, Springer, New York, 2007.

[9] J. Boltona, P. Gadera, H. Friguib, P. Torrione, Random set framework for multiple instance learning, Information Sciences 181 (2011) 2061–2070

[10] A. Collins, R.S. Michalski, The Logic of Plausible Reasoning: A Core Theory, Cognitive Science 13 (1989) 1–49.

[11] T.M. Cover, J.A. Thomas, Elements of information theory (1st Edition), Wiley-Interscience, New York, 2006.

[12] M. Delafontainea, P. Bogaerta, A.G. Cohnb, F. Witloxa, P. De Maeyer, N. Van de Weghe, Inferring additional knowledge from QTC_N relations, Information Sciences 181 (2011) 1573–1590

[13] J. Dian, A Meaning Based Information Theory - Informalogical Space: Basic Concepts and Convergence of Information Sequences, Information Sciences 180 (2010) 984–994.

[14] S.F. Ding, Z.Z. Shi, Studies on incidence pattern recognition based on information entropy, Journal of Information Sciences 31 (2005) 497-502

[15] C. Donga, C.S. Paty, Application of adaptive weights to intelligent information systems: An intelligent transportation system as a case study, Information Sciences 181 (2011) 5042–5052

[16] B. Galitsky, J.L. de la Rosa, Concept-based learning of human behavior for customer relationship management, Information Sciences 181 (2011) 2016–2035

[17] J. Goguen, K. Lin, G. Rosu, Circular Coinductive Rewriting, in: IEEE (Ed.), Proceedings of Automated Software Engineering '00, IEEE Press, Grenoble, 2000, pp. 123–131.

[18] S. Goldman, Information Theory, Dover, New York, 2005.

[19] I.R. Goodman, H. Nguyen, E. A. Walker, Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning, Elsevier, New York, 1991.

[20] Y. Guoa, Z. Shaoa, N. Hua, A cognitive interactionist sentence parser with simple recurrent networks, Information Sciences 180 (2010) 4695–4705

[21] J.Y. Halpern, Reasoning about uncertainty, MIT Press, Cambridge, 2003

[22] T. Hastie, R. Tibshirani, J. Friedman, The Elements of Statistical Learning, Springer, New York, 2001.

[23] T.M. Huang, V. Kecman, I. Kopriva, Kernel Based Algorithms for Mining Huge Data Sets, Supervised, Semi-supervised, and Unsupervised Learning, Springer-Verlag, Berlin, 2006.

[24] N. Kasabov, Evolving Connectionist Systems. The Knowledge Engineering Approach (2nd edition), Springer, New York, 2007.

[25] J.L. Kelley, General Topology, Springer-Verlag, New York, 1975

[26] G.J. Klir, U.H. St Clair, B. Yuan, Fuzzy set theory: foundations and applications, Prentice Hall, Englewood Cliffs, 1997.

[27] Y. Kodratoff, R.S. Michalski, Machine Learning: An Artificial Intelligence Approach (Volume III), Morgan Kaufmann, San Mateo, 1990.

[28] T.-T. Kuo1, S.-D. Lin, Learning-based concept-hierarchy refinement through exploiting topology, content and social information, Information Sciences 181 (2011) 2512–2528

[29] P. Lévy, From social computing to reflexive collective intelligence: The IEML research program, Information Sciences 180 (2010) 71–94

[30] Z.F. Li, Y.M. Liu, An approach to the management of uncertainty in expert systems, in: B.M. Ayyub (Ed.), Analysis and Management of Uncertainty: Theory and Applications, Elsevier, Amsterdam, 1991, pp. 133–140.

[31] H.W. Liu, G.J. Wang, Unified forms of fully implicational restriction methods for fuzzy reasoning, Information Sciences 177 (2007) 956–966

[32] G. Luger, W. Stubblefield, Artificial Intelligence: Structures and Strategies for Complex Problem Solving (5th ed.), The Benjamin/Cummings Publishing Company, Menlo Park, 2004.

[33] D.J.C. MacKay, Information Theory, Inference, and Learning Algorithms, Cambridge University Press, Cambridge, 2003.

[34] P. McCorduck, Machines Who Think: a personal inquiry into the history and prospects of artificial intelligence (2nd ed.), A K Peters, Natick, 2004.

[35] J. Mendel, Computing with words and its relationships with fuzzistics, Information Sciences 177 (2007) 988–1006

[36] R.S. Michalski, J.G. Carbonell, T.M. Mitchell, Machine Learning: An Artificial Intelligence Approach (Volume I), Tioga Publishing Company, Palo Alto, 1983.

[37] R.S. Michalski, J.G. Carbonell, T.M. Mitchell, Machine Learning: An Artificial Intelligence Approach (Volume II), Morgan Kaufmann, San Mateo, 1986.

[38] R.S. Michalski, G. Tecuci, Machine Learning: A Multistrategy Approach (Volume IV), Morgan Kaufmann, San Francisco, 1994.

[39] T. Mitchell, Machine Learning, McGraw-Hill, New York, 1997.

[40] Y. Neuman, A theory of meaning, Information Sciences 176 (2006) 1435–1449.

[41] N. Nilsson, Artificial Intelligence: A New Synthesis, Morgan Kaufmann Publishers, San Francisco, 1998.

[42] K.-M. Osei-Bryson, Towards supporting expert evaluation of clustering results using a data mining process model, Information Sciences 180 (2010) 414–431

[43] R.O. Parreirasa, P.Ya. Ekelb, J.S.C. Martinic, R.M. Palhares, A flexible consensus scheme for multicriteria group decision making under linguistic assessments, Information Sciences 180 (2010) 1075–1089

[44] W. Pedrycz, Fuzzy Sets Engineering, CRC Press, Boca Raton, 1995.

[45] W. Pedrycz, Computational Intelligence: An Introduction, CRC Press, Boca Raton, 1997.

[46] D. Poole, A. Mackworth, R. Goebel, Computational Intelligence: A Logical Approach, Oxford University Press, New York, 1998.

[47] A.L. Ralescu, D.A. Ralescu, Y. Yamakata, Inference by aggregation of evidence with applications to fuzzy probabilities, Information Sciences 177 (2007) 378–387

[48] N.J. Randon, J. Lawry, Classification and query evaluation using modelling with words, Information Sciences 176 (2006) 438–464

[49] S.J. Russell, P. Norvig, Artificial Intelligence: A Modern Approach (2nd ed.), Prentice Hall, Upper Saddle River, 2003.

[50] M.C. Schut, On model design for simulation of collective intelligence, Information Sciences 180 (2010) 132–155

[51] R. Vigo, Representational information: a new general notion and measure of information, Information Sciences 181 (2011) 4847–4859

[52] G.J. Wang, Formalized theory of general fuzzy reasoning, Information Sciences 160 (2004) 251–266

[53] L. Wanga, X. Liua, J. Cao, A new algebraic structure for formal concept analysis, Information Sciences 180 (2010) 4865–4876

[54] R. Yager, On some new classes of implication operators and their role in approximate reasoning, Information Sciences 167 (2004) 193–216

[55] R. Yager, A framework for reasoning with soft information, Information Sciences 180 (2010) 1390–1406

[56] L.A. Zadeh, Fuzzy Logic = Computing with Words, IEEE Transactions on Fuzzy Systems 2 (1996) 103–111.

[57] L.A. Zadeh, From Computing with Numbers to Computing with Words – From Manipulation of Measurements to Manipulation of Perceptions, IEEE Transactions on Circuits and Systems 45 (1999) 105–119.

[58] L.A. Zadeh, Outline of a computational theory of perceptions based on computing with words, in: N.K. Sinha, M.M. Gupta, L.A. Zadeh (Eds.), Soft Computing & Intelligent Systems: Theory and Applications, Academic Press, London, 2001, pp. 2–33.